POSTS AND TELECOMMUNICATIONS INSTITUTE OF TECHNOLOGY

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Chapter 1: MULTIVARIABLE CALCULUS

CALCULUS 2

Faculty of Fundamental Science 1

Hanoi - 2022

1.1.1. Set $\mathbb{R}^n$, distance, neighborhood, open set, closed set, bounded set

1. $\mathbb{R}^n = \{M(x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R},\ i = \overline{1, n}\}$.
2. Given $M(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $N(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The distance between a point M and N is denoted by $d(M, N)$, calculating by the formula
$$d(M, N) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$
3. Given $M(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\varepsilon > 0$. The set $\{M \in \mathbb{R}^n : d(M, M_0) < \varepsilon\}$ is called the ε- neighborhood of the point M_0.

4. Given $E \subset \mathbb{R}^n$. The point $M \in E$ is called the interior point of E if there exists a $\varepsilon > 0$ - some neighborhood of M that lies entirely in E.
5. The point $M \in \mathbb{R}^n$ is called the boundary point of E if every $\varepsilon > 0$ - neighborhood of M contains points belongs to E and points does not belong to E.
6. A set E is called open set if all its points are interior points, and closed if it contains its boundary.
7. A set E is called bounded set if there exists some closed sphere containing that set.

Example 1

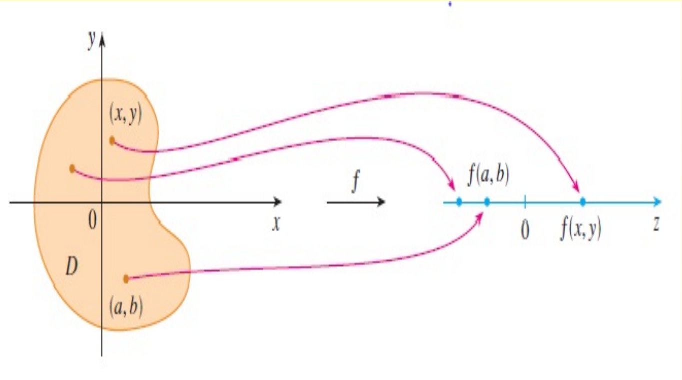
Given $M_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Set $\{M \in \mathbb{R}^n : \ d(M, M_0) < \varepsilon\}$ is an open set (called an open sphere with center M_0, radius ε).
Set $\{M \in \mathbb{R}^n : \ d(M, M_0) \leq \varepsilon\}$ is a closed set (called a closed sphere M_0, radius ε.

1.1.2. Functions of several variables

Definition

Given $\emptyset \neq D \subset \mathbb{R}^n$, a mapping $f : D \to \mathbb{R}$ such that $M(x_1, x_2, \cdots, x_n) \in D \mapsto z = f(M) = f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$ is called a function of n variables defined on D. Where, the D is called the **domain** of the function f and $x_1, x_2, \cdots, x_n$ are called independent variables.

Note: A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $z = f(x, y)$. The set D is the **domain** of f and its **range** is the set of values f that takes on, that is, $Im(f) = \{f(x, y) | (x, y) \in D\}$.



Hình 1.1: *Graph of the function* $z = f(x, y)$

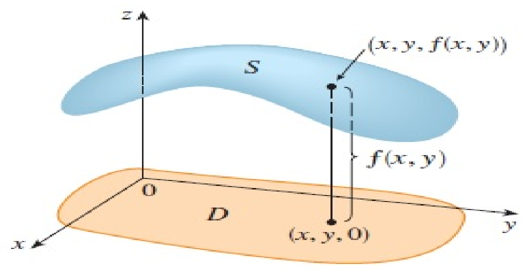
Example 2

Find the domains of the following functions and calculate $f(3,2)$.

a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$; b) $f(x,y) = x\ln(y^2 - x)$

Behavior of a function of two variables

If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.



Hình 1.2: *Graph of the function* $z = f(x, y)$

Example 3

Find the domain and range and sketch the graph of the following

1.2.1. Limits

Definition 1

The sequence of points $M_n(x_n, y_n)$ is said to approach the point $M_0(x_0, y_0)$, $n \to \infty$ if $\lim\limits_{n\to\infty} d(M_n, M_0) = 0$ or $\begin{cases} \lim\limits_{n\to\infty} x_n = x_0 \\ \lim\limits_{n\to\infty} y_n = y_0, \end{cases}$
denoted by $\lim\limits_{n\to\infty} M_n = M_0$ or $M_n \to M_0$ when $n \to \infty$.

Definition 2

Given the function $f(M) = f(x, y)$ defined on the domain D. The point $M_0(x_0, y_0)$ may or may not be in D. The function $f(M)$ has a limit $L \in \mathbb{R}$, $M \to M_0$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0 : (\forall M \in D),\ 0 < d(M, M_0) < \delta \Rightarrow |f(M) - L| < \varepsilon,$$

denoted by $\lim\limits_{M\to M_0} f(M) = L$ or $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L.$

Remark

If any small interval $(L-\varepsilon, L+\varepsilon)$ is given around L, we can find a disk D_δ with center (a,b) and radius $\delta > 0$ such that f maps all the points in D_δ (except possibly (a,b)) into the interval $(L-\varepsilon, L+\varepsilon)$.

Example 1

Prove that $\lim\limits_{(x,y)\to(1,2)} 2x+3y = 8$.

Definition 3

The function $f(M)$ has a limit of $L \in \mathbb{R},\ M \to M_0$ if for every sequence of points $\{M_n(x_n, y_n)\}$ are in the neighborhood of the point $M_0(x_0, y_0)$ such that $M_n \to M_0$, we have: $\lim\limits_{M\to M_0} f(M) = L$.

Example 2

Calculate $I = \lim_{(x,y)\to(2,1)} 3x - 5y.$

Remark

If $f(x,y) \to L_1$ as $(x,y) \to (x_0,y_0)$ along a path C_1 and $f(x,y) \to L_1$ as $(x,y) \to (x_0,y_0)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Example 3

Find the following limits

a) $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$;

b) $\lim_{(x,y)\to(1,2)} (x^3 - 2xy)$

c) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$;

d) $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$.

Definition 4

1. Given the function $f(M)$ defined on the domain D and $M_0 \in D$. The function $f(M)$ is called continuous at M_0 if $\lim\limits_{M \to M_0} f(M) = f(M_0)$.
2. if domain D is closed set and M_0 is the boundary of D, then the limit $\lim\limits_{M \to M_0} f(M) = f(M_0)$ where $M \in D$.
3. The function $f(M)$ is called continuous in the domain D if it is continuous at every point in D.

Example 4

Consider the continuity of the following function:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Definition

Let the function $z = f(x, y)$ define in D and $M_0(x_0, y_0) \in D$.

1) Suppose that only x is variable while keeping $y = y_0$, it means that y_0 is a constant. Then we are really considering a function of a single variable x, namely $f(x, y_0)$. If $f(x, y_0)$ has a derivative at x_0 then it is called the partial derivative of f with respect to variable x at M_0 and denote it by $z'_x(M_0) = f'_x(M_0)$ or $\dfrac{\partial z}{\partial x}(M_0) = \dfrac{\partial f}{\partial x}(M_0)$. Thus

$$\frac{\partial f}{\partial x}(M_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

2) Similarly, the partial derivative of with respect to the variable y at M_0, denoted by $z'_y(M_0) = f'_y(M_0)$ or $\dfrac{\partial z}{\partial y}(M_0) = \dfrac{\partial f}{\partial y}(M_0)$, is defined by $\dfrac{\partial f}{\partial y}(M_0) = \lim\limits_{\Delta y \to 0} \dfrac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$.

Example 1

For $f(x, y) = 2x^3 - 5x^2y + 3y^2 + 1$, find

$$\frac{\partial f}{\partial x}(x, y);\ \frac{\partial f}{\partial y}(x, y);\ \frac{\partial f}{\partial x}(2, 1);\ \frac{\partial f}{\partial y}(-1, 3).$$

Example 2

The productivity of an airplane-manufacturing company is given approximately by the Cobb–Douglas production function $f(x, y) = 40x^{0,3}y^{0,7}$, with the utilization of x units of labor and y units of capital.

a) Find $f'_x(x, y)$ and $f'_y(x, y)$.

b) If the company is currently using 1,500 units of labor and 4,500 units of capital, find the marginal productivity of labor and the marginal productivity of capital.

1.3.2 The chain rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

The chain rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example 3

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}(0)$.

The chain rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example 4

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$.

1.3.2 The chain rule

The chain rule (General version)

Suppose that u is a differentiable function of the n variables $x_1, x_2, ..., x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i},$$

for each $i = 1, 2, ..., m$.

Example 5

a) If $u = x^4y + y^2z^3$, where $x = rse^t$ and $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find $\dfrac{\partial u}{\partial s}$ when $r = 2, s = 1, t = 0$.

b) If $u = f(x^2 - y^2, y^2 - x^2)$ and f is differentiable, show that u satisfies the equation $y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$.

1.3.3 Implicit differentiation

Theorem 1

We suppose that an equation of the form $F(x, y) = 0$ defines implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x))$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the chain rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since both x and y are functions of x, we obtain $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$. But $\frac{dx}{dx} = 1$, so if $\frac{\partial F}{\partial y} \neq 0$ we solve for $\frac{dy}{dx}$ and obtain

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F'_x}{F'_y}.$$

Example 6

Find y' if $x^5 - y^5 = 5xy$.

1.3.3 Implicit differentiation

Theorem 2

we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the chain rule to differentiate the equation as follows $\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$. If $\frac{\partial F}{\partial x} \neq 0$, we solve for $\frac{\partial z}{\partial x}$ and obtain $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F'_x}{F'_z}$. Similarly, we obtain $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F'_y}{F'_z}$.

Example 7

Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $x^4 + y^3 + z^5 - 6xyz = 2$.

1.3.4 Total differentials

Definition 1.3.4

If $z = f(x, y)$, then f is differentiable at $M_0(x_0, y_0)$ if $\Delta f(M_0)$ can be expressed in the form

$$\Delta f(M_0) = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y + \varepsilon_1 \Delta x + +\varepsilon_2 \Delta y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 1.3.4

If the partial derivatives f'_x and f'_y exist near $M_0(x_0, y_0)$ and are continuous at M_0, then $f(x, y)$ is differentiable at M_0.

1.3.4 Total differentials

Definition 1.3.5

Assume the function f is differentiable at $M_0(x_0, y_0)$, the expression $f'_x(M_0)\Delta x + f'_y(M_0)\Delta y$ is called the total differential of f at $M_0(x_0, y_0)$, denoted by $df(M_0) = df(x_0, y_0)$. Thus,
$df(M_0) = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y$
Note: If we consider the functions $f(x, y) = x$ and $g(x, y) = y$ in $\mathbb{R}^2$, then $df(x, y) = dx = \Delta x$, $dg(x, y) = dy = \Delta y$. So the total differential of the function $f(x, y)$ at M_0 is also written as:

$$df(M_0) = f'_x(M_0)dx + f'_y(M_0)dy.$$

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Example 1

Given the function $f(x, y) = x^4 + 4xy^2 - 3x2y^3$. Calculate
a) $df(x, y)$; b) $df(1, 2)$ with $\Delta x = 0,1$; $\Delta x = 0,2$.

1.4.1 Higher derivatives

Definition 1.4.1

If $f(x, y)$ is a function of two variables, then its partial derivatives f'_x, f'_y are also function of two variables, so we can consider their partial derivatives $(f'_x)'_x$, $(f'_x)'_y$, $(f'_y)'_x$, $(f'_y)'_y$, which are called second order partial derivatives of $f(x, y)$. If $z = f(x, y)$, we use the following notation:

$$f''_{xx} = f''_{x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad f''_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y},$$

$$f''_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x}, \quad f''_{yy} = f''_{y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

Theorem 1.4.1 (Schwarz's theorem)

Suppose $f(x, y)$ is defined on a disk D that contains the point $M_0(x_0, y_0)$. If the functions $f"_{xy}$ and $f"_{yx}$ are both continuous on D, then $f''_{xy}(M_0) = f''_{yx}(M_0)$. Partial derivatives of order 3 or higher can also be defined. For instance $f'''_{xyy} = (f''_{xy})'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$.

Example 2

Given the function $f(x, y) = x^3 y - 3x^2 y^2 + 5y^4$.

a) Calculate first and second partial derivatives of $f(x, y)$.

b) Calculate $f^{(3)}_{x^2 y}$, $f^{(3)}_{xyx}$, $f^{(4)}_{x^2 y^2}$.

Definition 1.4.2

If $f(x, y)$ is a differentiable at (x, y) then $df = f'_x dx + f'_y dy$ is called a first-order differential of $f(x, y)$.
If $df(x, y)$ is differentiable at (x, y) then the total differential of $df(x, y)$ is called the second-order differential of $f(x, y)$, denoted by $d^2 f(x, y)$ defined by

$$d^2 f(x, y) = d(df(x, y)) = d(f'_x dx + f'_y dy).$$

Similarly, we have $d^n f(x, y) = d(d^{n-1} f(x, y))$, $n \in \mathbb{N}^*$. The second order differential formula is as follows:

$$d^2 f = d(df) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dy$$

$$= \frac{\partial^2 f}{\partial x^2}dx^2 + \left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x}\right)dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$$

.

1.4.2 Higher differentials

Definition 1.4.2

According to Schwarz theorem we have:

$$d^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2.$$

Example 3

Given the function $f(x, y) = x^2 y^3 + 2xy^2 - x^3 y^4$. Calculate
a) $d^2 f(x, y)$.
b)$d^2 f^(2, 1)$.

1.5.1 Directional derivatives

Definition 1.5.1

The directional derivative of f at $M_0(x_0, y_0, z_0)$ in the direction of a unit vector $\vec{l} = (a, b, c)$ is

$$\frac{\partial f}{\partial \vec{l}}(M_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + ah, z_0 + ah) - f((x_0, y_0, z_0))}{h},$$

if this limit exists.

Theorem 1.5.1

If f is a differentiable function of x, y and z, then f has a directional derivative in the direction of any unit vector $\vec{l} = (a, b, c)$ and

$$\frac{\partial f}{\partial \vec{l}}(x, y, z) = f'_x(x, y, z)a + f'_y(x, y, z)b + f'_z(x, y, z)c.$$

Example 1

Find the directional derivative $f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{l}$ is the unit vector given by the angle $\frac{\pi}{6}$. What is $\dfrac{\partial f}{\partial \vec{l}}(1,2)$?

1.5.2 Gradient vector

Definition 1.5.2 If f is a function of three variables x, y and z, then the gradient of f is the vector function $\overrightarrow{grad} f$ defined by

$$\overrightarrow{grad} f(x,y,z) = (f'_x, f'_y, f'_z) = f'_x \vec{i} + f'_y \vec{j} + f'_z \vec{k}.$$

Remark

With this notation for the gradient vector, we have

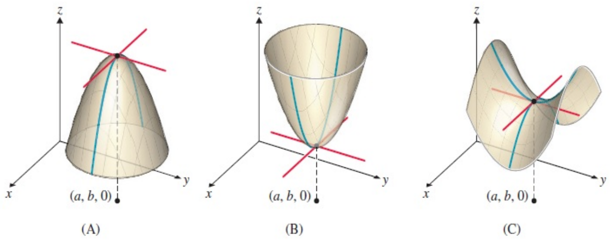
$$\frac{\partial f}{\partial \vec{l}}(x,y,z) = \overrightarrow{grad} f(x,y,z).\vec{l}$$

Definition 1.66.1

A function $f(M) = f(x, y)$ has a local maximum (local manimum) at $M_0(x_0, y_0) \in D$ if $f(M) \leq f(M_0)$ $(f(M) \geq f(M_0))$ when M is near M_0 (i.e $f(M) \leq f(M_0)(f(M) \geq f(M_0))$ for all points M in some disk with center M_0). The number $f(M_0)$ is called a **local maximum value (local manimum value)**. The maximum or minimum point are called the **critical point (or stationary point)**.

Note: If the inequalities in Definition 1.6.1 hold for all points $M(x, y)$ in the domain of $f(M)$, then $f(M)$ has an **absolute maximum (or absolute minimum)** at M_0.

Example 1.6.1



Hình 6.1: *Graph of the function $z = f(x, y)$ has extreme point*

Theorem 1.6.1 (Necessary condition)

If $f(x, y)$ has a local maximum or minimum at M_0 and the first order partial derivatives of $f(x, y)$ exist there, then

$$f'_x(M_0) = 0 \text{ and } f'_y(M_0) = 0.$$

Note

Theorem 1.6.1 gives us necessary (but not sufficient) conditions for $f(M_0)$ to be a local extremum. We find all points M_0 such that $f'_x(M_0) = 0$ and $f'_y(M_0) = 0$ and test these further to determine whether $f(M_0)$ is a local extremum or a saddle point. Points M_0 such that conditions $f'_x(M_0) = 0$ and $f'_y(M_0) = 0$, or if one of these partial derivatives does not exist are called **critical points.**

Theorem 1.6.2 (Second-Derivative Test for Local Extrema (Sufficient condition)

Suppose the second partial derivatives of $f(M) = f(x, y)$ are con tinuous on a disk with center M_0, and suppose that the point M_0 is a critical point of $f(M)$. Put
$A = f"_{xx}(M_0),\ B = f"_{xy}(M_0),\ C = f"_{yy}(M_0)$, and $\Delta = B^2 - AC$.

1. if If $\Delta < 0$ and $A < 0$, then $f(M_0)$ is a local maximum.
2. if If $\Delta < 0$ and $A > 0$, then $f(M_0)$ is a local minimum.
3. if If $\Delta > 0$ then $f(x, y)$ has a saddle point at M_0 (i.e then $f(M_0)$ is not local maximum or minimum).

Remark

1. In case (3) the point $M_0(x_0, y_0)$ is called a **saddle point** of f.
2. If $\Delta = 0$, the test gives no information: f could have a local maximum or local minimum at $M_0(x_0, y_0)$, or M_0 could be a saddle point of f.
3. To remember the formula for Δ, it's helpful to write it as a determinant $\Delta = f"_{xx} f"_{yy} - (f"_{xy})^2$.

Example 1

Find the local maximum and minimum values of the following functions:

a) $f(x, y) = -x^2 - y^2 + 6x + 8y - 21$.

b) $f(x, y) = x^3 + y^3 - 3xy + 2$.

c) $f(x, y) = x^4 + y^4 - 4xy + 1$; d) $f(x, y) = x^3 + y^2$.

1.7 Conditional extremes

Definition

A point $M_0(x_0, y_0) \in \mathbb{R}^2$ is called the maximum (minimum) point of the function $z = f(x, y)$ subject to a constraint (or side condition) $\varphi(x, y) = 0$ if it satisfies $\varphi(M_0) = 0$ and there exists a small enough neighborhood of M_0 on the constraint curve $\varphi(x, y) = 0$ such that $f(M) \leq f(M_0)$ (or $f(M) \geq f(M_0)$.

Maxima using Lagrange multipliers

This problem is one of a general class of problems of the form:
Maximize or minimize $f(x, y) \quad (a)$
subject to $\varphi(x, y) = 0 \quad (b)$
Now to the method: We form a new function F, using functions f and φ in equations (a) and (b), as follows:
$F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y), \quad (c)$
Here, λ (the Greek lowercase letter lambda) is called a **Lagrange multiplier**.

1.7 Conditional extremes

Theorem (Method of Lagrange multipliers for functions of two variables)

Any local maxima or minima of the function $z = f(x, y)$ subject to the constraint $\varphi(x, y) = 0$ will be among those points (x_0, y_0) for which (x_0, y_0, λ_0) is a solution of the system $F'_x(x, y, \lambda) = 0$, $F'_y(x, y, \lambda) = 0$, and $F'_\lambda(x, y, \lambda) = 0 \quad (d)$, where $F(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$ provided that all the partial derivatives exist.

Example 1

Find maximize

a) $f(x, y) = xy$ subject to $3x + y - 720 = 0$.

b) $f(x, y) = 6 - 4x - 3y$ subject to $x^2 + y^2 - 1 = 0$.

Example 2

Find maximize $f(x, y, z) = x + y + z^2$ subject to $z - x = 1$ and $y - xz = 1$.

1.8 Maximum and minimum values

For a function f of one variable the extreme value theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a closed set in $\mathbb{R}^2$ is one that contains all its boundary points (A boundary point of D is a point such that every disk with center contains points in D and also points not in D).

Definition 1.8.1

If $f(M)$ is continuous on a closed, bounded set D in$\mathbb{R}^n (n \in N^*)$, then $f(M)$ attains an absolute maximum value $f(M_1)$ and an absolute minimum value $f(M_2)$ at some points M_1 and M_2 in$D$.

1.8 Absolute maximum and minimum values

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Step 1. Find the values of f at the critical points of f in D.
2. Step 2. Find the extreme values of f on the boundary of D.
3. Step 3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

Find the absolute maximum and minimum values of the following functions:

a) $f(x,y) = x^2 - 2xy + 2y$, where $D = \{(x,y)|0 \leq x \leq 3,\ 0 \leq y \leq 2\}$.

b) $f(x,y) = x^2 + y^2 - 2x^2y$, where $D = \{(x,y)|x^2 + y^2 \leq 1\}$.

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Chapter 2: MULTIPLE INTEGRALS

CALCULUS 2

Faculty of Fundamental Science 1

Hanoi - 2022

1. 2.1 Integral depends on a parameter
2. 2.2 Double integrals
3. 2.3. Triple integrals

2.1.1 Definite integral depends on a parameter

Definition 2.1

Let $f : [a, b] \times [c, d] \to \mathbb{R}$, for each fixed $y \in [c, d]$, the function $f(x, y)$ is integrable over $[a, b]$ on the x variable. Then, the following function $F : [a, b] \to \mathbb{R}$ as

$$F(y) = \int_a^b f(x, y)dx$$

is called an integral depending on a parameter. The function $F(y)$ has the following properties:

Theorem 2.1

If the function $f(x, y)$ is continuous on $[a, b] \times [c, d]$ then $F(y)$ is continuous on $[c, d]$.

2.1.1 Definite integral depends on a parameter

Remark 2.1

If $f(x,y)$ is continuous on $[a,b]\times[c,d]$, and $\alpha(y),\beta(y)$ are continuous on $[c,d]$ with $a\le\alpha(y),\beta(y)\le b$, $\forall y\in[c,d]$ then $F(y)=\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx$ is continuous on $[c,d]$.

Example 2.1

Let the function $f(x)$ be continuous on $[0,1]$. Prove that

$$F(y)=\int\limits_0^1 \frac{y^2 f(x)}{x^2+y^2}dx$$

is continuous on $(0,+\infty)$.

2.1.1 Definite integral depends on a parameter

Theorem 2.2

If $f(x,y)$ and $f'_y(x,y)$ are continuous on $[a,b]\times[c,d]$, then $F(y)$ differentiable on $[c,d]$ and $F'(y)=\int\limits_a^b f'_y(x,y)dx$.

Theorem 2.3 (Leibniz's Theorem)

Let $f(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ be continuous functions on $[a,b]\times[c,d]$, and $\alpha(y),\beta(y)$ differentiable functions on $[c,d]$ with image on $[a,b]$, that is, $\alpha(y),\beta(y):[a,b]\to[c,d]$, $\forall x\in[\alpha(y),\beta(y)]\subset[c,d]$. We define $F(y)=\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx$, then $F(y)$ is differentiable on $[c,d]$ and

$$F'(y)=f(\beta(y),y).\beta'(y)-f(\alpha(y),y).\alpha'(y)+\int\limits_a^b f'_y(x,y)dx.$$

Example 2.2

Calculate the derivative of the following function

$$F(y) = \int_0^1 \arctan\frac{x}{y}\,dx,\ \ y > 0.$$

Theorem 2.4

Let $f(x,y)$ be integrable over $[a,b]\times[c,d]$, then $F(y)$ is differentiable on $[c,d]$ and

$$\int_c^d F(y)dy = \int_c^d \left(\int_a^b f(x,y)dx\right)dy = \int_a^b \left(\int_c^d f(x,y)dy\right)dx.$$

Example 2.3

Calculating integrals

$$I = \int_0^1 \frac{x^b - x^a}{\ln x} dx, \ b > a > 0.$$

2.1.2 Improper integral depends on parameter

Definition 2.2

1. Let $f : D := [a, +\infty) \times [c, d] \to \mathbb{R}$, for each fixed $y \in [c, d]$ the function $f(x, y)$ is integrable over $[a, +\infty)$ on the x variable. Then, the function

$$F(y) = \int_a^{+\infty} f(x, y)dx$$

is called an improper integral of depending on parameter y.

2. The function $F(y)$ is called uniformly converge for each $y \in [c, d]$, if $\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon, y) > 0,\ \forall b \geq n_0 \Rightarrow \left| \int_b^{+\infty} f(x, y)dx \right| < \varepsilon.$

3. The function $F(y)$ is called uniformly converge on the interval $[c, d]$, if $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^*,\ \forall b \geq n_0 \Rightarrow \left| \int_b^{+\infty} f(x, y)dx \right| < \varepsilon,\ \forall y \in [c, d].$

Theorem 2.5 (Weierstrass' theorem)

If the function $\int\limits_a^{+\infty} h(x)dx$ converges and $|f(x,y)| \leq h(x),\ \forall (x,y) \in D$ then the function $F(y)$ is uniformly converge on $[c,d]$.

Example 2.4

Prove that

$$\int\limits_1^{+\infty} \frac{\cos(x+2y)}{x^2+y^2}dx$$

is continuous on $\mathbb{R}$.

2.1.2 Improper integral depends on parameter

Theorem 2.6

If the function $f(x, y)$ is continuous on $[a, +\infty) \times [c, d]$ and the function $F(y)$ is uniformly convergent on $[c, d]$ then then $F(y)$ is continuous on $[c, d]$.

Example 2.5

Prove that

$$\int_{1}^{+\infty} \frac{x}{2 + x^y} dx$$

is continuous on $(2, +\infty)$.

2.1.2 Improper integral depends on parameter

Theorem 2.7

If the function $f(x,y)$ is continuous on $[a,+\infty)\times[c,d]$ and the function $F(y)$ is uniformly convergent on $[c,d]$ then $F(y)$ is differentiable on $[c,d]$ and

$$\int_c^d F(y)dy = \int_c^d \left(\int_a^{+\infty} f(x,y)dx \right) dy = \int_a^{+\infty} \left(\int_c^d f(x,y)dy \right) dx.$$

Example 2.6

Given $b > a > 0$, calculate the following integral:

$$I = \int_0^{+\infty} \frac{e^{-ax} - e^{-bx}}{x} dx.$$

2.1.2 Improper integral depends on parameter

Theorem 2.8

Let $f(x,y)$ be define on the D satisfying the following assumptions:
1) the function $f(x,y)$ is continuous in the variable x on $[a,+\infty)$ for each $y \in [c,d]$,
2) a function $f'_y(x,y)$ is continuous in domain D,
3) the funtion $F(y)$ converges for each $y \in [c,d]$,
4) an integral $\int\limits_a^{+\infty} f'_y(x,y)dx$ converges uniformly on the $[c,d]$

Then, $F'(y) = \int\limits_a^{+\infty} f'_y(x,y)dx$.

Example 2.6

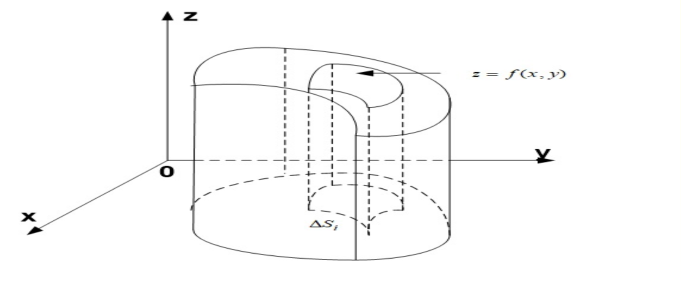
Find the derivative of the function
$F(y) = \int\limits_0^{+\infty} \frac{1-\cos xy}{xe^{2x}} dx, y \in (0,+\infty)$, and find the function $F(y)$.

2.2.1 Definition of double integrals

Problem

Calculate the volume of the bounded domain V is given by:

- (Oxy) is a plane.
- The axis Oz and the standard curve L is the boundary of the finite closed domain $D \subset (Oxy)$.
- The curved surface is the graph of a function of two variables $z = f(x, y), (x, y) \in D$.



Definition

Let the function $z = f(x, y)$ define on a closed domain $D \subset R^2$.

+ Divide D into n small regions $S_1, S_2, ..., S_n$ by a grid of curves, name and area the domains denoted by $\Delta S_i (i = 1, \ldots, n)$ and denoted d_i is the diameter of the second piece S_i as follows:

$$d_i = \max\{AB : A \in S_i, B \in S_i\}. \; Set \; \Delta_n = \max\{d_1, d_2, ..., d_n\}.$$

+ Choose an arbitrary point $M_i \in S_i$. Then $\sigma_n = \sum_{i=1}^{n} f(M_i)\Delta S_{\mathrm{i}}$ is called the *sum of the integrals* of $f(x, y)$ on the domain D. If $I = \lim\limits_{\Delta_n \to 0} \sigma_n$ exists and does not depend on the partition S_i and choosing $M_i \in S_i$, then number I is called the double integrals of $f(x, y)$ on the domain D and it is denoted by

$$\iint\limits_D f(x, y)dxdy. \quad \text{So} \quad \iint\limits_D f(x, y)dxdy = \lim_{\Delta_n \to 0} \sum_{i=1}^{n} f(M_i)\Delta S_i.$$

Note

1) Since the double integral does not depend on the division of the domain D should be able to divide D by a grid of lines parallel to the coordinate axes Ox, Oy . Then $dS = dx \cdot dy$. Therefore, the double integral is denoted by

$$I = \iint_D f(x, y) dxdy$$

2) Like definite integrals, the symbol of a variable that is double integrated does not change the double integral, that is,

$$\iint\limits_D f(x, y) dxdy = \iint\limits_D f(u, v) dudv = I.$$

2.2.2 Integral conditions of double integrals

- If the function $f(x, y)$ is integrable over the domain D then $f(x, y)$ is bounded on the domain D (necessary condition of integrable function).
- If the function $f(x, y)$ is continuous on the D , more general: If the function $f(x, y)$ is only a discontinuity of type 1 on domain D, then it is integrable on the domain D.

Let $f(x,y), g(x,y)$ be integrable on D. Then, we have

1) $\iint_D [f(x,y) \pm g(x,y)]dxdy = \iint_D f(x,y)dxdy \pm \iint_D g(x,y)dxdy$.
2) $\iint_D k \cdot f(x,y)dxdy = k \iint_D f(x,y)dxdy$, $\forall k$.
3) If $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$ then

$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$$

2.2.3 Properties of double integrals

4) If $f(x,y) \leq g(x,y), \forall (x,y) \in D$ then

$$\iint\limits_D f(x,y)dxdy \leq \iint_D g(x,y)dxdy$$

5) , If $f(x,y)$ is integral on D then $|f(x,y)|$ is also integrable on D and

$$\left|\iint_D f(x,y)dxdy\right| \leq \iint_D |f(x,y)|dxdy$$

6) If $f(x,y)$ is integral on D and satisfies $m \leq f(x,y) \leq M, \forall (x,y) \in D$ then

$$mS \leq \iint_D f(x,y)dxdy \leq MS.$$

where S is the area of the domain D.

2.2.4 Double integrals over rectangles

Theorem 2.2.1 (Fubini's Theorem)

Let $f(x,y)$ be continuous on $D = [a,b] \times [c,d]$ (the domain D is a rectangular domain). We have

$$\iint\limits_D f(x,y)dxdy = \int\limits_a^b dx\Big(\int\limits_c^d f(x,y)dy\Big) = \int\limits_c^d dy\Big(\int\limits_a^b f(x,y)dx\Big).$$

Example 2.2.1

Calculating integrals $I = \iint\limits_D (2x+y)dxdy$, where $D = [1,2] \times [0,2]$.

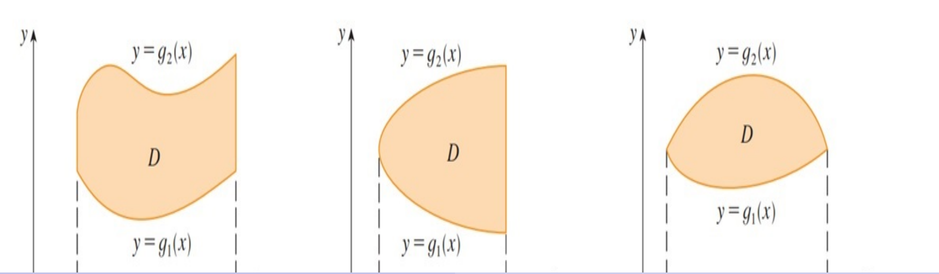
Example 2.2.2

Calculating integrals $I = \iint\limits_D xy^2dxdy$, where $D = [0,2] \times [0,3]$.

Type I regions

A plane region D is said to be of type I, if
$D = \left\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\right\}$. Then,

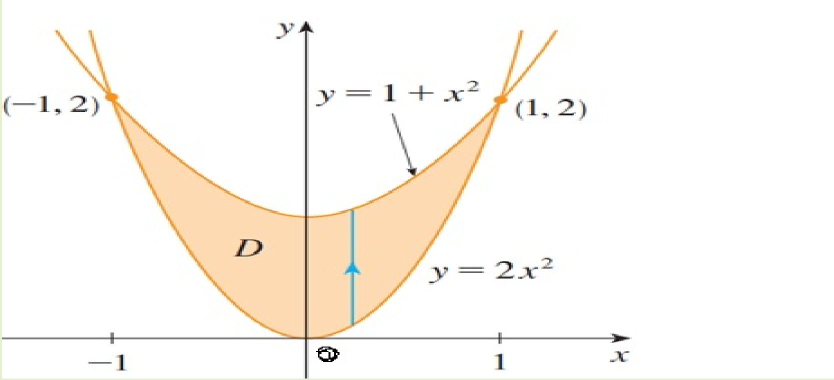
$$\iint\limits_D f(x,y)dxdy = \int\limits_a^b dx \int\limits_{g_1(x)}^{g_2(x)} f(x,y)dy.$$



Example 2.2.3

Evaluate $I = \iint\limits_{D} (x^2 + 2y)dxdy$, where D is a region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

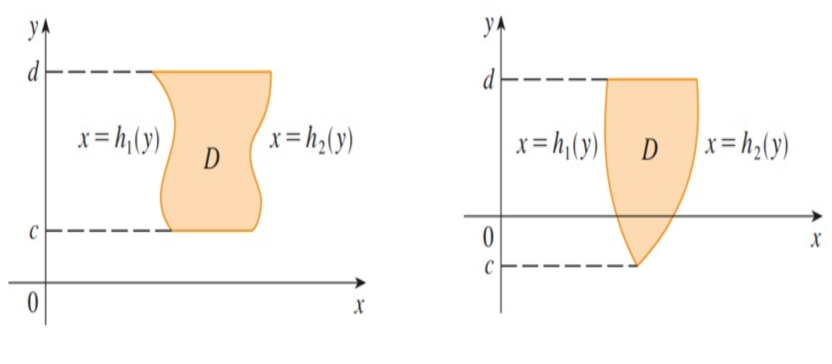
Solution



Type II regions

A plane region D is said to be of type II. if
$D = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$. Then,

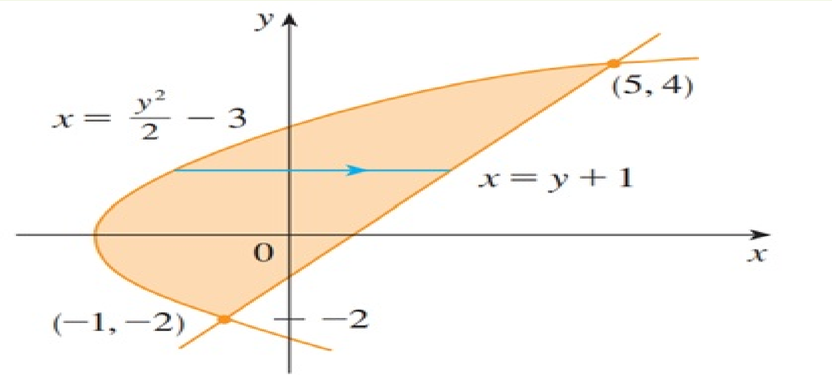
$$\iint_D f(x,y)dxdy = \int_c^d dx \int_{h_1(y)}^{h_2(y)} f(x,y)dx.$$



Example 2.2.4

Evaluate $I = \iint_D xy\,dxdy$, where D is a region bounded by the parabolas $y = x - 1$ and $y^2 = 2x + 6$.

Solution



Example 2.2.5

Change the order of integral in double integrals

$$a) I = \int_0^2 dx \int_x^{2x} f(x,y)dy.$$

.

$$b) J = \int_{-2}^{6} dy \int_{-\frac{y^2}{2}-1}^{2-y} f(x,y)dx.$$

$$c) K = \int_0^1 dx \int_x^{\sqrt{2-x^2}} f(x,y)dy.$$

2.2.5 Change of variables in double integrals

Let the function $f(x,y)$ be continuous on the domain $D \subset (Oxy)$ and assume the transformation $(x,y) \to (u,v) : \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ satisfying the condition

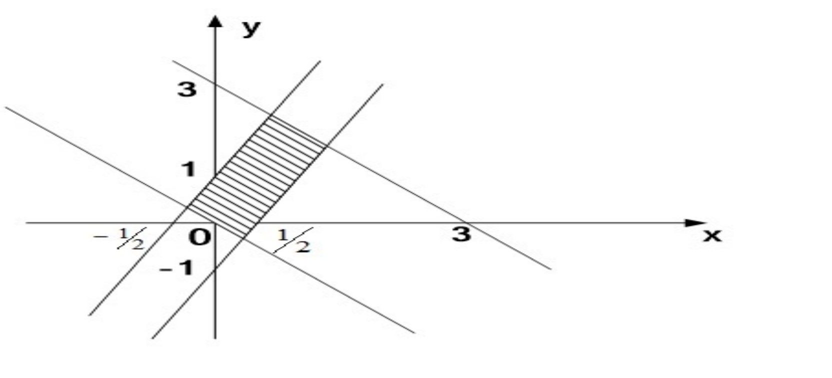
- The above transformation is a bijective from Δ to the domain D or $(x,y) \in D \Leftrightarrow (u,v) \in \Delta$.
- The $x(u,v), y(u,v)$ are the continuous partial derivatives on the domain $\Delta \subset (O'uv)$.
- The Jacobi determinant is $\frac{D(x,y)}{D(u,v)} \neq 0$ on the domain Δ (or just zero at some isolated point) then

$$I = \iint_D f(x,y)dxdy = \iint_\Delta f[x(u,v), y(u,v)] \cdot \left| \frac{D(x,y)}{D(u,v)} \right| dudv.$$

Example 2.2.6

Calculating integrals $I = \iint_D (x+y)dxdy$, where D is
$y = -x, y = -x+3, y = 2x-1, y = 2x+1.$

Solution



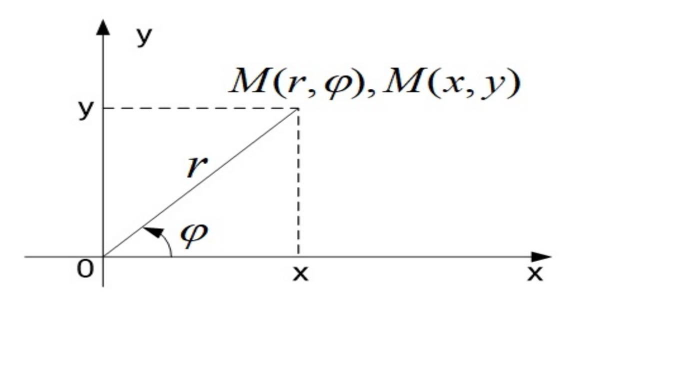
2.2.5 Change of variables in double integrals

Example 2.2.7

Calculating integrals $I = \iint_D x^3 dxdy$, where D is
$y = \frac{1}{x}, y = \frac{2}{x}, y = x^2, y = \frac{x^2}{2}$.

a. polar coordinate system

A polar coordinates are set of real numbers (r, φ) so that
$r = |\overrightarrow{OM}|, \varphi = (Ox, \overrightarrow{OM})$



b. calculate the double integrals

Set $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow D \to \Delta$. Condition: $\left\{(r,\varphi) \mid \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \end{cases}\right\}$

$$\frac{D(x,y)}{D(r,\varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

Then the double integrals in polar coordinates has the form

$$\iint\limits_D f(x,y)dxdy = \iint\limits_\Delta f(r\cos\varphi, r\sin\varphi)rdrd\varphi.$$

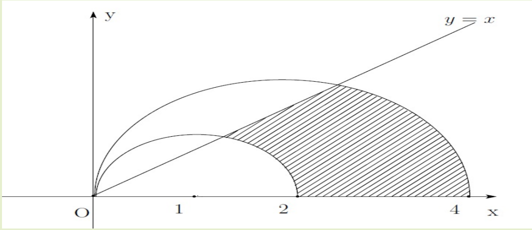
1. Finding area of a plane regions

If $f(x,y)=1,\ \forall(x,y)\in D$ then the measure of the area of the domain D is calculated according to the formula $S_D=\iint_D dxdy$.

Example 2.2.9

Calculate the area of the domain D given by

$$D=\left\{(x,y):(x-1)^2+y^2=1,\ (x-2)^2+y^2=4,\ y=x,\ y=0\right\}.$$



2.2.7 Applications of double integrals

2. Computing volumes

If $f(x, y) \geq 0,\ \forall (x, y) \in D$ then the volume of the curved cylinder bounded by the function graph is calculated by the formula

$$V = \iint_D f(x, y)dxdy.$$

Example 2.2.10

Calculate the volume of the figure V given by the following faces

$$z = x^2 + y^2,\ y = x^2,\ y = 1,\ z = 0.$$

2.2.7 Applications of double integrals

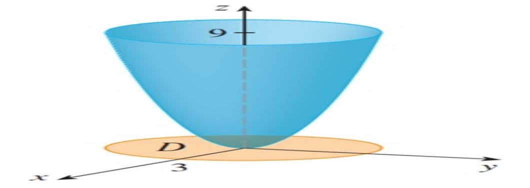
3. Surface area

For surface $(S): z = f(x,y),\ (x,y) \in D$ has partial derivatives f'_x, f'_y exist and are continuous on domain D. Then the surface area of S is defined as

$$A(S) = \iint_D \sqrt{1 + f_x'^2 + f_y'^2}\,dxdy.$$

Example 2.2.11

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



4. Density mass

Suppose the lamina occupies a region D of the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. The density mass of the lamina is

$$m = \iint\limits_D \rho(x, y)dxdy.$$

If the plate is homogenous, that is $\rho(x, y) = \text{const}, \forall(x, y) \in D$, choose $\rho(x, y) = 1, \forall(x, y) \in D$ then the mass of the plate D is calculated by the formula $m = \iint_D dxdy = S_D$

5. Center of mass

$$x_G = \frac{1}{m}\iint_D x\rho(x, y)dxdy,\ y_G = \frac{1}{m}\iint_D y\rho(x, y)dxdy,$$

where $m = \iint_D \rho(x, y)dxdy$.

2.2.7 Applications of double integrals

4. Density mass

Suppose the lamina occupies a region D of the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. The density mass of the lamina is

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where $m = \iint_D \rho(x, y)dxdy$.

6. Moment of inertia

According to the definition of the moment of inertia of the particle about the Ox, Oy -axis and the origin O, we have

$$I_{Ox} = my^2;\ I_{Oy} = mx^2; \quad I_O = m\left(x^2 + y^2\right)$$

Moment of inertia of the plate about the axes Ox, Oy and the origin O are

$$I_{Ox} = \iint_D y^2 \rho(x,y)dxdy; I_{Oy} = \iint_D x^2 \rho(x,y)dxdy;$$

$$I_O = \iint_D \left(x^2 + y^2\right)\rho(x,y)dxdy.$$

Problem

Calculate the mass of the non-homogeneous body V, given that the density is $\rho = \rho(x, y, z), (x, y, z) \in V$. Similar to the double integral, we divide V arbitrarily into n parts that do not step on each other. Name and volume of the parts $\Delta V_i (i = \overline{1, n})$. Choose an arbitrary point $P_i\left(x_i, y_i, z_i\right) \in \Delta V_i$ and the $d_i, (i = \overline{1, n})$ are diameters of $\Delta V_i (i = \overline{1, n})$. We have

$$m \approx \sum_{i=1}^{n} \rho\left(P_i\right) \Delta V_i = \sum_{i=1}^{n} \rho\left(x_i, y_i, z_i\right) \Delta V_i$$

The mass of the object is

$$m = \lim_{\max d_i \to 0} \sum_{i=1}^{n} \rho\left(x_i, y_i, z_i\right) \Delta V_i$$

2.3.1 Definition of triple integrals

Definition

Let the function $f(x,y,z)$ define on the domain $V \subset R^3$.

- Divide V into n pieces, name and volume of the piece are $\Delta V_i (i = \overline{1,n})$, the piece diameter symbol ΔV_i is d_i, $i = \overline{1,n}$.
- Choose an arbitrary point $P_i\,(x_i, y_i, z_i) \in \Delta V_{i,}(i = \overline{1,n})$.
- The totals $I_n = \sum_{i=1}^{n} f\,(x_i, y_i, z_i)\,\Delta V_i$ is called the sum of integrals the triple of the function $f(x,y,z)$ taken over the domain V corresponds to a fraction plan and points $P_i \in \Delta V_i, (i = \overline{1,n})$.. When $n \to \infty$ such that $\max d_i \to 0$, we get I_n converges to $I \in \mathbb{R}$ regardless of the partition ΔV_i and how point $P_i\,(x_i, y_i, z_i) \in \Delta V_i$ is chosen, the number I is called a triple integral of $f(x,y,z)$ over the region V and denoted by

$$\iiint\limits_V f(x,y,z)dxdydz = \lim_{\max_i d_i \to 0} \sum_{i=0}^{n} f\,(x_i, y_i, z_i)\,\Delta V_i$$

2.3.1 Definition of triple integrals

Note

- Like the double integrals, the volume factor dV is replaced by $dxdvdz$ and then the triple integral is usually denoted by

$$I = \iiint\limits_V f(x,y,z)dxdydz$$

- Similar to the double integrals, triple integrals do not depend on the notation of the variable being integrated

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_V f(u,v,w)dudvdw$$

- If the function f(x, y, z) is continuous on the closed, bounded domain $V \in \mathbb{R}^3$, then it's integrable on V.
- The integral conditions and properties of triple integrals are similar to the double integrals.

2.3.2 Triple integrals on the rectangular box

Theorem 2.3.1 (Fubini's Theorem)

If $f(x,y,z)$ is continuous on the rectangular box $V=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint\limits_V f(x,y,z)dxdydz=\int_a^b\int_c^d\int_r^s f(x,y,z)dxdydz.$$

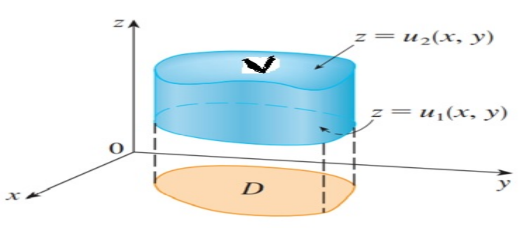
Example 2.3.1

Calculating integrals $I=\iiint\limits_V xyz^2dxdydz$, where

$$V=[0,1]\times[-1,2]\times[0,3].$$

2.3.2 The triple integral over a general bounded region

The region of type I



A solid region V is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,
$V = \{(x,y,z) : (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$, then

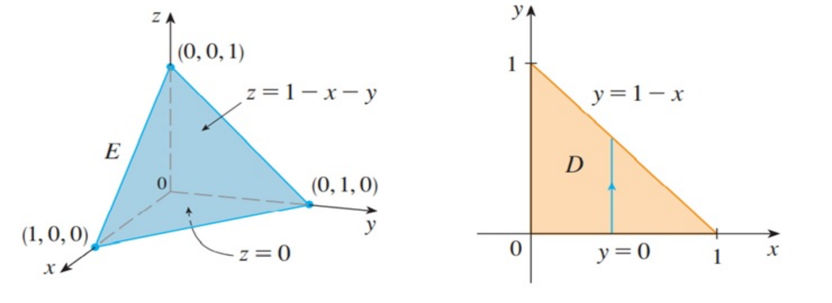
$$I = \iiint_V f(x,y,z)dxdydz = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right] dxdy.$$

where D is the projection of V onto xy-plane, $z = u_1(x, y)$ is the lower surface and $z = u_2(x, y)$ is the upper surface.

Example 2.3.2

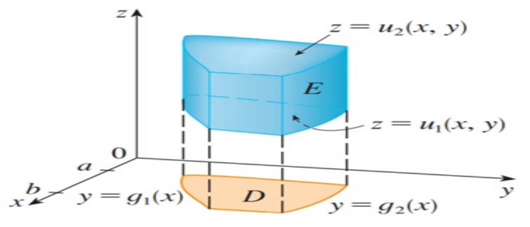
Evaluate $I = \iiint\limits_V z dxdydz$, where V is the solid tetrahedron bounded by the four planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

Solution



The region of type II

If the projection D of V onto the xy-plane is of type II plane region, then
$V = \{(x, y, z) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ u_1(x, y) \leq z \leq u_2(x, y)\}$,
and we have



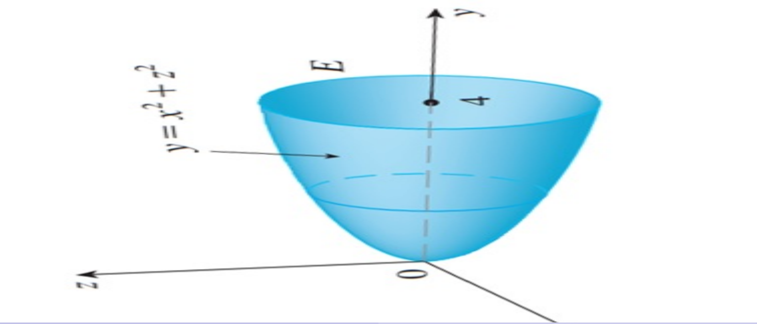
$$\iiint\limits_V f(x, y, z) dxdydz = \int\limits_a^b dx \int\limits_{g_1(x)}^{g_2(x)} dy \int\limits_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz.$$

where D is the projection of V onto xy-plane, $z = u_1(x, y)$ is the lower surface and $z = u_2(x, y)$ is the upper surface.

Example 2.3.3

Evaluate $\iiint\limits_V \sqrt{x^2 + z^2}\,dxdydz$, where V is the solid tetrahedron bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution



2.3.3 Change of variables in triple integrals

For the function $f(x, y, z)$ to be continuous on the domain $V \subset Oxyz$ and and assume the transformation

$$(x, y, z) \to (u, v.w) : \begin{cases} x = x(u, v), w) \\ y = y(u, v, w), \quad (u, v, w) \in \Omega \\ z = z(u, v, w) \end{cases}$$ satisfy the

conditions

- The above transformation is a bijective from Ω to the domain V or $(x, y, z) \in V \Leftrightarrow (u, v, w) \in \Omega$.
- The $x(u, v, w), y(u, v, w), z(u, v, w)$ are the continuous partial derivatives on the domain $\Omega \subset (O'uvw)$.
- The Jacobi determinant is $J = \frac{D(x,y,z)}{D(u,v,w)} \neq 0$ on the domain Ω, then

$$\iiint_V f(x, y, z)dxdydz = \iiint\limits_{\Omega} f[x(u, v, w), y(u, v, w), z(u, v, w)]|J|du$$

Example 2.3.4

Evaluate $\iint\limits_V (x+y)(x-z)dxdydz$, where V is the bounded domain by the planes
$x+y=0,\ x+y=1;\ y+z=1,\ y+z=2;\ x+y-z=2,\ x+y-z=3..$

Solution

Set

$$u=x+y, v=y+z, w=x+y-z$$

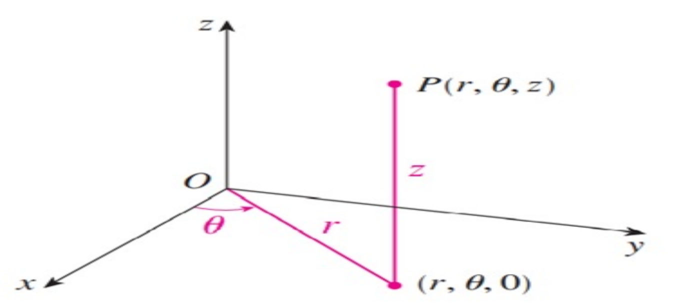
$$0\le u\le 1, 1\le v\le 2, 2\le w\le 3$$

$$\frac{D(u,v,\mathrm{w})}{D(x,y,z)}=-1\Rightarrow\frac{D(x,y,z)}{D(u,v,\mathrm{w})}=-1,(x+y)(x-z)=u(u-v)$$

$$I=\iiint_\Omega u(u-v)|-1|dudvdw=\int_0^1 udu\int_1^2(u-v)dv\int_2^3 dw=-\frac{5}{12}.$$

Cylindrical coordinates

In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z) , where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane. The connections between cylindrical coordinates and rectangular coordinates are



Evaluating triple integrals with cylindrical coordinates

Set $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$ then $V \to \Omega$. Conditions: $\begin{cases} r \geq 0 \\ 0 \leq \varphi < 2\pi \\ -\infty < z < +\infty \end{cases}$.

The Jacobi determinant of the functions x, y, z in terms of r, φ, z are

$$J = \frac{D(x,y,z)}{D(r,\varphi,z)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

The formula for triple integration in cylindrical coordinates is

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_\Omega f(r\cos\varphi, r\sin\varphi, z)rdrd\varphi dz.$$

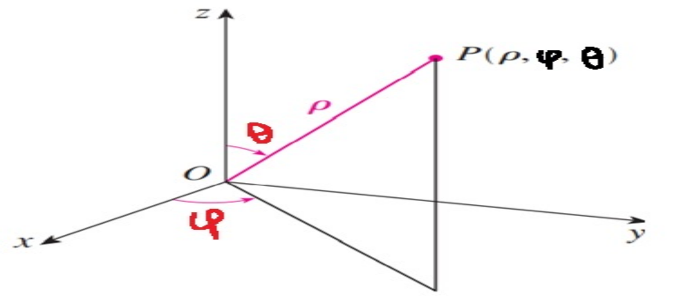
Example 2.3.5

Evaluate $I = \iiint\limits_V (x^2 + y^2 + 3z^2)dxdydz$, where

$$V = \{(x, y, z) : \sqrt{x^2 + y^2} \leq z \leq 2\}.$$

Spherical coordinates

The spherical coordinates of a point P in space are (ρ, φ, θ), where ρ is the distance from P to the origin, φ is the same angle as in cylindrical coordinates, and θ is the angle between the positive z-axis and the line segment OP. Note that $\rho \geq 0,\ 0 \leq \theta \leq \pi$.



2.3.5 Triple integrals in spherical coordinates

Evaluating triple integrals with spherical coordinates

Set $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$ then $V \to \Omega$. Condition: $\begin{cases} r \geq 0 \\ 0 \leq \varphi < 2\pi \\ 0 \leq \theta \leq 2\pi \end{cases}$.

The Jacobi determinant of the functions x, y, z in terms of r, φ, z are

$$J = \frac{D(x, y, z)}{D(r, \varphi, \theta)} = r^2 |\sin\varphi|.$$

The formula for triple integration in spherical coordinates is

$$I = \iiint\limits_{\Omega} f(r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\theta) r^2 |\sin\varphi| dr d\varphi d\theta.$$

2.3.5 Triple integrals in spherical coordinates

Example 2.3.6

Evaluate $I = \iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz$, where

a) V is the unit ball.

b) $V = \{(x,y,z) : x^2+y^2+z^2 \le 4\}$.

c) $V = \{(x,y,z) : 0 \le z \le \sqrt{4-x^2-y^2}\}$.

2.3.6 Applications of triple integrals

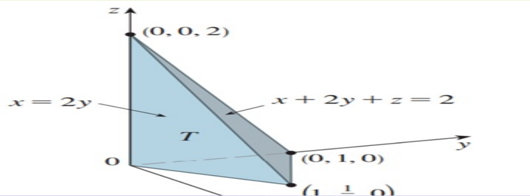
1. Volume

If $f(x, y, z) = 1$ for all point in V. Then the triple integral does represent the volume of V

$$V = \iiint\limits_{V} dV = \iiint\limits_{V} dxdydz.$$

Example 2.3.7

Use triple integral to find the volume of the tetrahedron V bounded by the planes $x + 2y + z = 2, x = 2y, x = 0$, and $z = 0$.



2. Mass of a solid object

If the density function of a solid object that occupies the region V is $\rho(x, y, z)$ (in units of mass per unit volume) at any given point (x, y, z), then its mass is

$$m = \iiint\limits_V \rho(x, y, z)dV = \iiint\limits_V \rho(x, y, z)dxdydz.$$

3. Moments

Its moments about the three coordinate planes are

$$I_{Ox} = \iiint\limits_V (y^2 + z^2)\rho(x,y,z)dV, \quad I_{Oy} = \iiint\limits_V (x^2 + z^2)\rho(x,y,z)dV$$

$$I_{Oz} = \iiint\limits_V (x^2 + y^2)\rho(x,y,z)dV, \quad I_O = \iiint\limits_V (x^2 + y^2 + z^2)\rho(x,y,z)dV.$$

4. Center of mass

The center of mass is located at the point G, where

$$x_G = \frac{1}{m}\iiint_V x\rho(x,y,z)dV, \quad y_G = \frac{1}{m}\iiint_V y\rho(x,y,z)dV$$

$$z_G = \frac{1}{m}\iiint_V z\rho(x,y,z)dV, \qquad m = \iiint_V \rho(x,y,z)dV.$$

Example 2.3.8

Find the center of mass (if the density is constant, the center of mass is called the centroid) of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0$, and $x = 1$.

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Chapter 3: LINE AND SURFACE INTEGRALS

CALCULUS 2

Faculty of Fundamental Science 1

Hanoi - 2022

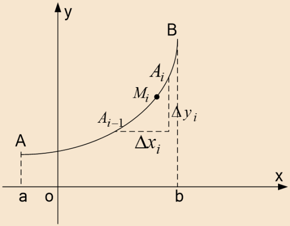
I. Line integral of type 1

1. Definition

Let the function f(x,y) define on a plane arc $\overset{\frown}{AB}$.

Divide P the arc $\overset{\frown}{AB}$ by n points

$$A_0 \equiv A, A_1, \ldots, A_{i-1}, A_i, \ldots, A_n \equiv B$$



Set

ΔS_i, $i = \overline{1,n}$ is called the arc length of $\overset{\frown}{A_{i-1}A_i}$.

Choose the

arbitrary points $M_i\,(x_i, y_i) \in \overset{\frown}{A_{i-1}A_i}$, $(i = \overline{1,n})$.

Then, $I_n = f(M_1)\Delta S_1 + ... + f(M_n)\Delta S_n = \sum_{i=1}^{n} f\,(x_i, y_i)\,\Delta S_i$

is called the sum of first-order

line integrals of the function $f(x,y)$ on arc $\overset{\frown}{AB}$.

I. Line integral of type 1

Consider the limit $I = \lim\limits_{\Delta_P \to 0} I_n$ where $\Delta_P = \max\{\Delta S_1, ..., \Delta S_n\}$. If I does not depend on the division of arc $\overset{\frown}{AB}$ and the choice of $M_i\,(x_i, y_i) \in \overset{\frown}{A_{i-1}A_i}, (i = \overline{1, n})$, then the number I is called the first-order line integral of $f(x, y)$ along arc $\overset{\frown}{AB}$ and the symbol $\int\limits_{\overset{\frown}{AB}} f(x, y)ds$.

So $I = \lim\limits_{\max \Delta S_i \to 0} \sum\limits_{i=1}^{n} f\,(x_i, y_i)\,\Delta S_i = \int\limits_{\overset{\frown}{AB}} f(x, y)ds$, where ds denotes the length factor of the arc or the differential of arc.

If the function $f(x, y, z)$ is integrable on arc $\overset{\frown}{AB} \subset \mathrm{R}^3$ then the first-order line integral of f(x,y,z) on arc AB denoted is

$$I = \int\limits_{\overset{\frown}{AB}} f(x, y, z)ds.$$

I.Line integral of type 1

- The arc $\overset{\frown}{AB}$ is called smooth if its tangent is variable continuous.
- The arc $\overset{\frown}{AB}$ is called a segmented smooth arc if arc $\overset{\frown}{AB}$ can be divided into a finite number of smooth arcs.
- It can be proved: If arc $\overset{\frown}{AB}$ is smooth or smooth each segment and f(x,y) is continuous on arc $\overset{\frown}{AB}$, then $f(x,y)$ is integrable on arc $\overset{\frown}{AB}$.
- The first-order line product has the same properties as the product definite stool

$$\int\limits_{\overset{\frown}{AB}} (\alpha f + \beta g)(x,y)ds = \alpha \int\limits_{\overset{\frown}{AB}} f(x,y)ds + \beta \int\limits_{\overset{\frown}{AB}} g(x,y)ds.$$

$$\int\limits_{\overset{\frown}{AC}} f(x,y)ds = \int\limits_{\overset{\frown}{AB}} f(x,y)ds + \int\limits_{\overset{\frown}{BC}} f(x,y)ds, \quad \forall B \in \overset{\frown}{AC}.$$

Remark 1

a) From the above definition, we see the direction of arc $\overset{\frown}{AB}$ plays no role because I_n does not depend on the direction of arc AB. So

$$\int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_{\overset{\frown}{BA}} f(x,y)ds$$

b) If l is the length of arc $\overset{\frown}{AB}$, then $l = \int\limits_{\overset{\frown}{AB}} ds$

Remark 1

c) If a material wire has arc $\overset{\frown}{AB}$ and mass density is $\rho(x,y)$, then the mass of the material wire is calculated according to the formula

$$m = \int\limits_{\overset{\frown}{AB}} \rho(x,y)ds.$$

The center of mass of the wire with density function $\rho(x,y)$ is located at the point $G(x_G, y_G)$, where

$$x_G = \frac{1}{m}\int\limits_{\overset{\frown}{AB}} x\rho(x,y)ds, \;\; y_G = \frac{1}{m}\int\limits_{\overset{\frown}{AB}} y\rho(x,y)ds.$$

2. The formula for first-order line integral

a. The arc $\overset{\frown}{AB}$ has the general form:

Case 1: Let $\overset{\frown}{AB}$ be smooth segmented arc of the form $y = y(x),\ x \in [a, b]$ and the function $f(x, y)$ is continuous on $\overset{\frown}{AB}$. Then

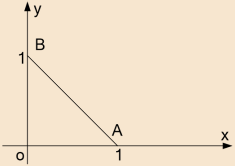
$$I = \int\limits_{\overset{\frown}{AB}} f(x, y)ds = \int\limits_a^b f(x, y(x))\sqrt{1 + y'^2(x)}\, dx \quad (3.1)$$

Case 2: Let $\overset{\frown}{AB}$ be smooth segmented arc of the form $x = x(y),\ y \in [c, d]$ and the function $f(x, y)$ is continuous on $\overset{\frown}{AB}$. Then

$$I = \int\limits_{\overset{\frown}{AB}} f(x, y)ds = \int\limits_c^d f(x(y), y)\sqrt{1 + x'^2(y)}\ dy \quad (3.2)$$

Example 1.

Calculate $\int\limits_C (x+y)ds$, where C is the boundary of the triangle with points $O(0,0), A(1,0), B(0,1)$.



Solution

$$\int\limits_C = \int\limits_{\overline{OA}} + \int\limits_{\overline{AB}} + \int\limits_{\overline{BO}} .$$

The segment $\overline{OA}$ has the equation $y = 0, 0 \le x \le 1$

$$\int\limits_{\overline{OA}} (x+y)ds = \int\limits_0^1 x\sqrt{1+0}dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}.$$

Continuity example 1.

The arc $\overset{\frown}{AB}$ has the equation $y = 1 - x, 0 \le x \le 1$

$$\Rightarrow \int\limits_{\overset{\frown}{AB}} (x+y)ds = \int\limits_0^1 1\sqrt{1+1}dx = \sqrt{2}$$

The segment $\overline{BO}$ has the equation $x = 0, 0 \le y \le 1$

$$\int\limits_{\overline{BO}} (x+y)ds = \int\limits_0^1 y\sqrt{1+0}dy = \frac{1}{2}y^2\Big|_0^1 = \frac{1}{2}$$

$$\Rightarrow \int\limits_C (x+y)ds = 1 + \sqrt{2}.$$

2. The formula for first-order line integrals

b. The arc $\overset{\frown}{AB}$ has parametric form in the plane

Let $\overset{\frown}{AB}$ be smooth segmented arc of the form

$$\overset{\frown}{AB}:\begin{cases} x = x(t) \\ y = y(t) \end{cases}, t_1 \leq t \leq t_2$$

and the function $f(x, y)$ is continuous on $\overset{\frown}{AB}$.Then

$$I = \int\limits_{\overset{\frown}{AB}} f(x, y)ds = \int\limits_{t_1}^{t_2} f[x(t), y(t)]\sqrt{x'^2(t) + y'^2(t)}dt$$

Note: The curves in space

$$\overset{\frown}{AB} \subset \mathbb{R}^3 : \begin{cases} x = x(t) \\ y = y(t), t_1 \le t \le t_2 \\ z = z(t) \end{cases}$$

$$\int\limits_{\overset{\frown}{AB}} f(x,y,z)ds = \int\limits_{t_1}^{t_2} f(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt.$$

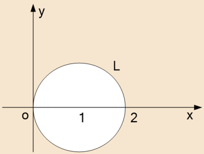
c. The curve in polar coordinates

$$\overset{\frown}{AB} : r = r(\varphi), \varphi_1 \le \varphi \le \varphi_2 \Rightarrow x'^2(\varphi) + y'^2(\varphi) = r^2(\varphi) + r'^2(\varphi)$$

$$I \int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_{\varphi_1}^{\varphi_2} f[r(\varphi)\cos\varphi, r(\varphi)\sin\varphi]\sqrt{r^2(\varphi)+r'^2(\varphi)}d\varphi$$

Example 2.

Calculating
$I = \int\limits_L \sqrt{x^2+y^2}ds$, where L is the circle $x^2+y^2=2x$.



The equation
of the line L in polar coordinates has the form

$$r = 2\cos\varphi, -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

$$I = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\varphi\sqrt{4\cos^2\varphi + 4\sin^2\varphi}d\varphi = 8\int\limits_0^{\frac{\pi}{2}}\cos\varphi d\varphi = 8\sin\varphi\Big|_0^{\frac{\pi}{2}} = 8$$

It is possible to integrate as a parameter $\begin{cases} x = 1+\cos t \\ y = \sin t \end{cases}, 0 \le t \le 2\pi$

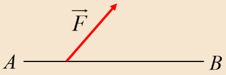
$$I = \int\limits_0^{2\pi}\sqrt{(1+\cos t)^2+\sin^2 t}dt = \int\limits_0^{2\pi}\sqrt{2+2\cos t}dt = \int\limits_0^{2\pi}\sqrt{4\cos^2\frac{t}{2}}dt = 8.$$

II. Line integral of type 2

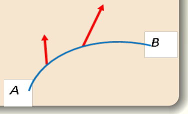
1. Problem: Calculate the power of the transformed force

A power produced by force $\vec{F}$ move on the arc L from A to B is

$$W = \vec{F} \cdot \overrightarrow{AB}$$



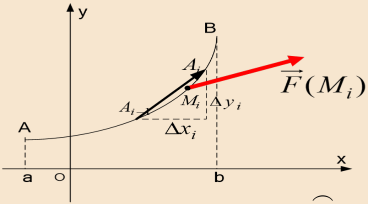
Calculate the power W of the force $\vec{F}(M)$ born while moving on the arc AB from point A to point B



$$\vec{F}(M) = P(M)\vec{i} + Q(M)\vec{j} = (P, Q); M \in AB$$

II. Line integral of type 2

- Divide arc $\overset{\frown}{AB}$ by n the points $A_0, A_1, \ldots, A_n$. Let the coordinates of the vector $\overrightarrow{A_{i-1}A_i}$ be $(\Delta x_i, \Delta y_i)$ and the arc length $\overset{\frown}{A_{i-1}A_i}$ is $\Delta s_i,\ i = \overline{1,\ n}$.



- Choose the arbitrary points $M_i\,(x_i, y_i) \in \overset{\frown}{A_{i-1}A_i},\ i = \overline{1,\ n}$.
- So that the power W of the force produced from A to B on arc AB approximately $W \approx \sum\limits_{i=1}^{n} \left[P\,(x_i, y_i)\,\Delta x_i + Q\,(x_i, y_i)\,\Delta y_i\right]$

$$\Rightarrow W = \lim_{\max \Delta S_i \to 0} \sum^{n} \left[P\,(x_i, y_i)\,\Delta x_i + Q\,(x_i, y_i)\,\Delta y_i\right].$$

II. Line integral of type 2

When $n \to \infty$ so that $\max \Delta s_i \to 0\,(\max \Delta x_i \to 0, \max \Delta y_i \to 0)$ that I_n converge to a number I regardless of the division of the arc L and the arbitrary choice $M_i \in \overset{\frown}{A_{i-1}A_i}$ then the number I is called a line integral of the second type of functions $P(x,y), Q(x,y)$ along arc L go from A. Denote by

$$\int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy.$$

Thus

$$\int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_i \to 0}} \sum_{i=1}^{n} \left[P\left(x_i, y_i\right) \Delta x_i + Q\left(x_i, y_i\right) \Delta y_i\right].$$

II. Line integral of type 2

Remark

- Unlike the first-order line integral, in the first-order line integral two, the direction of integration of L is important If integrating along arc $\overset{\frown}{AB}$ going from B to A, the vectors silk $\overrightarrow{A_{i-1}A_i}$ change direction. So the sum of the integrals will change sign, so

$$\int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy = -\int\limits_{\overset{\frown}{BA}} P(x,y)dx + Q(x,y)dy.$$

- The power produced by force $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ for the point to move from A to B along the arc $\overset{\frown}{AB}$ will be

$$W = \int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy.$$

Remark

- If the $\overset{\frown}{AB}$ is a curve in space and the functions $P(x,y,z), Q(x,y,z), R(x,y,z)$ define on the arc $\overset{\frown}{AB}$ then product the second-order segmentation of these three functions is also denoted by

$$\int\limits_{\overset{\frown}{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz.$$

- Let L be a plane curve and closed curve.we use the convention that the positive orientation of a simple closed curve L refers to a single counterclockwise traversal of L. That is a person walking along L in that direction will see the domain bounded by the L closest to me is on the left. The integral taken in the positive direction is denoted by $\oint\limits_{L} P(x,y)dx + Q(x,y)dy$.

Remark

- If two functions $P(x,y), Q(x,y)$ are continuous on smooth arc $\overset{\frown}{AB}$ or segmented smooth, then there exists a line integral of the second-order

$$I = \int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy$$

Line integrals of the second-order have the same properties as definite integrals.

Note:

- Sum, difference, multiply a number of the second-order line integrals

$$\int\limits_{\overset{\frown}{AC}} Pdx + Qdy = \int\limits_{\overset{\frown}{AB}} Pdx + Qdy + \int\limits_{\overset{\frown}{BC}} Pdx + Qdy, \quad \forall B \in \overset{\frown}{AC}.$$

3.The formula for calculating line integrals of type 2

Let the two functions $P(x,y), Q(x,y)$ be continuous on smooth arc $\overset{\frown}{AB}$ is given by the parametric equation $\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t : t_A \to t_B$. Then

$$\int\limits_{\overset{\frown}{AB}} Pdx + Qdy = \int\limits_{t_A}^{t_B} \left[P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t) \right] dt,$$

where arc $\overset{\frown}{AB}$ is planar given by an equation of the form

$$y = y(x); A\left(x_A, y\left(x_A\right)\right), B = \left(x_B, y\left(x_B\right)\right)$$

$$I = \int P(x,y)dx + Q(x,y)dy = \int^{b} \left[P(x, y(x)) + Q(x, y(x))y'(x) \right] dx.$$

Example 1.

Calculating work done by force $\vec{F} = -y\vec{i} + x\vec{j}$ born along the road ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in its positive orientation.

Solution

Parametric equation of the ellipse

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, 0 \leq t \leq 2\pi$$

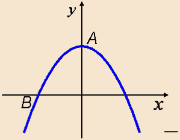
$$A = \int\limits_L xdy - ydx = \int\limits_0^{2\pi} (a\cos t \cdot b\cos t + b\sin t \cdot a\sin t)dt = ab\int\limits_0^{2\pi} dt = 2\pi ab$$

II. Line integral of type 2

Example 2.

Calculating $I = \int\limits_L \left(2xy - x^2\right) dx + \left(x + y^2\right) dy$

where L is the arc of the parabola
$y = 1 - x^2$ go from point A(0,1) to point B (-1,0)



$$y = 1 - x^2 \Rightarrow dy = -2xdx$$

$$I = \int_0^{-1} \left[2x\left(1 - x^2\right) - x^2 + \left(x + 1 - 2x^2 + x^4\right)(-2x)\right] dx$$

$$= \int_0^{-1} \left(-2x^5 + 2x^3 - 3x^2\right) dx$$

$$= \left(-\frac{1}{3}x^6 + \frac{1}{2}x^4 - x^3\right)\Big|_0^{-1} = -\frac{1}{3} + \frac{1}{2} + 1 = \frac{7}{6}$$

3. Green's formula

Theorem 1. (Green's formula)

Let ∂D be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by ∂D. Then,

$$\iint\limits_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint\limits_{\partial D} Pdx + Qdy$$

Example 3.

By using Green's formula to evaluate
$I = \oint\limits_{\partial D} (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$, with ∂D is circle $x^2 + y^2 = 4$.

Example 4.

Calculate $J = \int\limits_{C} \left(x \arctan x + y^2\right) dx + \left(x + 2yx + y^2 e^{-y^3}\right) dy$, with C is given by the equation $OA : x^2 + y^2 = 2x, y \geq 0$ going from origin to $A(0, 2)$.

4. Equivalent propositions for line integrals of type 2

Theorem 2.

Assume that the functions $P(x,y), Q(x,y)$ are continuous with the derivatives their first-order exclusivity in the simple domain D, then the following propositions are equivalent

- $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \forall (x,y) \in D$
- $\oint\limits_L Pdx + Qdy = 0$, where L is any closed curve in the domain D.
- The $\int\limits_{\overset{\frown}{AB}} Pdx + Qdy$, depends only on 2 points A and B but it does't depend on arc type $\overset{\frown}{AB}$ on the domain D.
- the $Pdx + Qdy$ is the total differential of the function some $u(x,y)$ on the domain D.

II. Line integral of type 2

Corollary 1

If $du(x,y) = Pdx + Qdy$ in the D domain, then

$$\int_{\overset{\frown}{AB}} Pdx + Qdy = u(B) - u(A)$$

Corollary 2

If $Pdx + Qdy$ is the total differential of the function $u(x,y)$ on the domain $D \in\subset \mathbb{R}^2$ then the function $u(x,y)$ is given by the formula:

$u(x,y) = \int\limits_{x_0}^{x} P(x,y)dx + \int\limits_{y_0}^{y} Q(x_0,y)dy + C$ or

$u(x,y) = \int\limits_{x_0}^{x} P(x,y_0)dx + \int\limits_{y_0}^{y} Q(x,y)dy + C$, where

$M_0(x_0,y_0),\ M(x,y) \in D$.

Example 5.

Prove that the expression

$$\left(x^2 - 2xy^2 + 3\right)dx + \left(y^2 - 2x^2y + 4y - 5\right)dy$$

is the total differential of the function $u(x, y)$ on the R^2 and find the function $u(x, y)$.

$$\frac{\partial Q}{\partial x} = -4xy = \frac{\partial P}{\partial y}, \forall (x, y) \in \mathrm{R}^2$$

$$\Rightarrow \exists u(x,y) : \begin{cases} \frac{\partial u}{\partial x} = P(x,y) = x^2 - 2xy^2 + 3 \\ \frac{\partial u}{\partial y} = Q(x,y) = y^2 - 2x^2y + 4y - 5 \end{cases} \Rightarrow \begin{cases} u = \frac{x^3}{3} - x^2y^2 + \\ \Rightarrow \frac{\partial u}{\partial y} = -2x^2y \end{cases}$$

$$\Rightarrow f'(y) = y^2 + 4y - 5 \Rightarrow f(y) = \frac{y^3}{3} + 2y^2 - 5y + C$$

$$\Rightarrow u = \frac{x^3}{3} - x^2y^2 + 3x + \frac{y^3}{3} + 2y^2 - 5y + C.$$

Example 6.

Evaluate $I = \int\limits_{\overset{\frown}{AB}} \frac{xdy - ydx}{x^2+y^2}$, where $A(1,1), B(\sqrt{3},3)$.

a) The arc $\overset{\frown}{AB}$ is given by the equation: $y = x^2, 1 \leq x \leq \sqrt{3}$.

b) Let the arc $\overset{\frown}{AB}$ make the segment AB a closed curve that does not cover the origin.

III. Surface integral of type 1

1. Definition of surface integral of the of type 1

Let the function $f(M) = f(x, y, z)$ be define on the curved surface S.
- Divide the surface S into n pieces that do not step on each other, name and the symbol for the area, the diameter of the i-th piece is $\Delta S_i, d_i; i = \overline{1, n}$.
- Choose the arbitrary points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i, i = \overline{1, n}$.
- The totals $I_n = \sum_{i=1}^n f\left(M_i\right) \Delta S_i$ is called the total surface integral of type one for a division of the surface S and choice of the points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i, i = \overline{1, n}$.

If when $n \to \infty$ such that $\max d_i \to 0$ that I_n converges to the number I depends on the division of the surface S and selects the points $M_i \in \Delta S_i$, then the number I is called the first-order surface integral of $f(M)$ on the surface S, denoted by $\iint\limits_S f(x, y, z) dS$.

So $I = \iint\limits_S f(x, y, z) dS = \lim\limits_{\max d_i \to 0} \sum\limits_{j=1}^{n} f\left(x_i, y_i, z_i\right) \Delta S_i$.

2. Integral conditions and properties of surface integral of type 1.

- If the surface S is smooth (the surface S has a normal variation continuous) or piecewise smooth (dividing S into a finite number of smooth surfaces and the function $f(x, y, z)$ is continuous or piecewise on the surface S, then there exists a first-order surface integral of that function on S.
- A face integral of the first kind has the same properties as a double integral

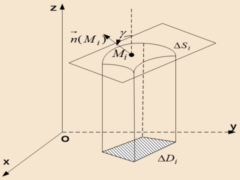
$$\iint\limits_S (\alpha f + \beta g) dS = \alpha \iint\limits_S f dS + \beta \iint\limits_S g dS.$$

$$\iint\limits_S f dS = \iint\limits_{S_1} f dS + \iint\limits_{S_2} f dS; \; S = S_1 \cup S_2, S_1 \cap S_2 = \varnothing.$$

3. Therem (How to calculate surface integrals of type 1)

Let the function $f(x, y, z)$ be continuity on a smooth surface S given by equation $z = z(x, y), (x, y) \in D$. Then

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,z(x,y))\sqrt{1 + z'_{x^2}(x,y) + z'_{y^2}(x,y)}dxdy$$



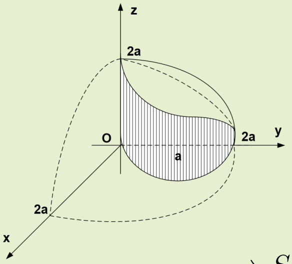
$$0 = dF(x,y,z) = F'_x dx + F'_y dy + F' z dz$$

$$\Rightarrow \cos\gamma = \pm \frac{F_z}{\sqrt{F_x'^2 + F_y'^2 + F_z'^2}} = \pm \frac{1}{\sqrt{1 + z_x'^2 + z_y'^2}}$$

$$I_n = \sum_{i=1}^{n} f(M_i)\Delta S_i \approx \sum_{i=1}^{n} f(x_i, y_i, z_i)\sqrt{1 + z_x'^2 + z_y'^2} \cdot \Delta D_i; \Delta S_i \approx \frac{\Delta D_i}{|\cos\gamma_i|}.$$

Example 1.

Calculate the area of the upper part of the sphere $x^2 + y^2 + z^2 = 4a^2$ inside the cylinder $x^2 + y^2 \leq 2ay, a > 0$ The upper sphere has the equation $z = \sqrt{4a^2 - x^2 - y^2}$.



$$D : x^2 + (y-a)^2 \leq a^2, x \geq 0$$

$$S = \iint\limits_S dS = \iint\limits_D \sqrt{1 + z_x'^2 + z'^2} dxdy$$

$$z'x = -\frac{x}{z}, z'y = -\frac{y}{z} \Rightarrow \sqrt{1 + z_x'^2 + z'^2} = \frac{2a}{|z|}$$

$$\Rightarrow S = \iint\limits_D \frac{-2a}{\sqrt{4a^2 - x^2 - y^2}} dxdy.$$

III. Surface integral of type 1

Converting to polar coordinates, we get

$$S = 2a \int_0^{\pi} d\varphi \int_0^{2a\sin\varphi} \frac{r\,dr}{\sqrt{4a^2 - r^2}} = 8a^2\left(\frac{\pi}{2} - 1\right).$$

Remark

- The case of surface S is given by the equation $y = y(z, x)$ or $x = x(y, z)$ then we have to project S onto the Ozx or Oyz to find the corresponding double integral.
- In the case of a curved surface of any shape, we must divide it into a number finite parts that satisfy the above theorem, then apply the formula.

4. Applications

- From the definition, we get the formula for surface area curvature S thanks to surface integrals of the first-order surface is $S = \iint\limits_S dS$.
- If S is a material surface, the mass density function is $\rho(x, y, z)$ then the mass of that material surface will be

$$m = \iint\limits_S \rho(x, y, z) dS.$$

- Formula for determining the center of gravity of a curved surface

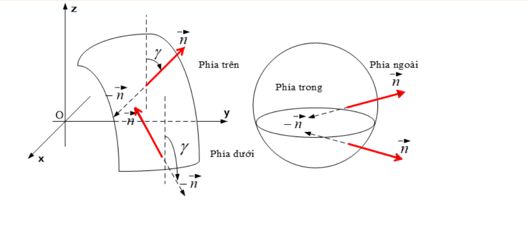
$$x_G = \frac{1}{m} \iint\limits_S x\rho(M) dS, y_G = \frac{1}{m} \iint\limits_S y\rho(M) dS, z_G = \frac{1}{m} \iint\limits_S z\rho(M) dS.$$

1. Oriented surfaces

- A smooth S-curve is called an oriented if the normal vector unit line $\vec{n}(M)$ completely determined at every $M \in S$ (can subtract the boundary of S) and transform continuously as M runs over S.
- The set $\vec{n}(M), \forall M \in S$ of a oriented curved surface define one side of the surface. Because $-\vec{n}(M)$ is also a normal vector, so the oriented surface always has two sides.

1. Oriented surfaces

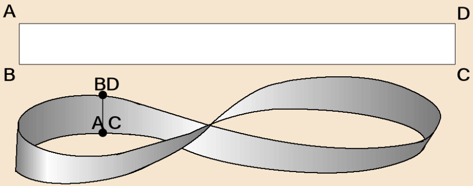
- When the S-curve is not closed and oriented, one usually used from above and below to indicate a specified direction determined by $\vec{n}(M)$. The top of the S-face is the side that $\vec{n}(M)$ with angle Oz axis pointed, and the bottom is the side $\vec{n}(M)$ with Oz axis obtuse angle.
- When the closed S-curve is oriented, one uses the side in and out to describe the specified direction.



IV. Surface integral of type 2

1. Oriented surfaces

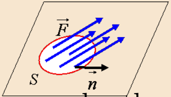
- Outside is the side $\vec{n}(M)$ outward of the object V surrounded by the S-curve, inside is the opposite side.
- There is a curved surface that cannot be oriented, for example, the following surface is called Möbius strip.



IV. Surface integral of type 2

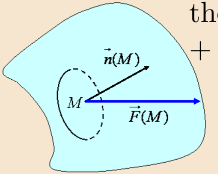
2. Calculate the flux of the vector field through a surface

The Flux of the vector field silk is constant across the plane



$$\Phi = S \cdot \vec{F} \cdot \vec{n}$$

Divide the surface S into n pieces that do not step on each other, name and the symbol for the area, the diameter of the i-th piece is $\Delta S_i, d_i; i = \overline{1, n}$



+ Choose the arbitrary points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i, i = \overline{1, n}$.
+ Throughput approx

$$\Phi \approx \Phi_n = \sum_{i=1}^{n} \Delta S_i \cdot \vec{F}\left(M_i\right) \cdot \vec{n}\left(M_i\right)$$

IV. Surface integral of type 2

Suppose

$$\vec{F}\left(M_i\right)=\left(P\left(M_i\right);Q\left(M_i\right);R\left(M_i\right)\right),\vec{n}\left(M_i\right)=\left(\cos\alpha_i;\cos\beta_i;\cos\gamma_i\right)$$

$$\begin{aligned}\Phi_n&=\sum_{i=1}^{n}\Delta S_i\vec{F}\left(M_i\right)\cdots\vec{n}\left(M_i\right)\\&=\sum_{i=1}^{n}\left(P\left(M_i\right)\cos\alpha_i+Q\left(M_i\right)\cos\beta_i+R\left(M_i\right)\cos\gamma_i\right)\Delta S_i\end{aligned}$$

The flux of the vector field $\vec{F}$ through the S -curve in the direction $\vec{n}$

$$\Phi=\lim_{\max d_i\to 0}\sum_{i=1}^{n}\left(P\left(M_i\right)\cos\alpha_i+Q\left(M_i\right)\cos\beta_i+R\left(M_i\right)\cos\gamma_i\right)\Delta S_i$$

3. Definition of the surface integral of type 2

Let the surface S oriented along the normal vector $\vec{n}(M)$ and three functions $P(x, y, z), Q(x, y, z), R(x, y, z)$ determined on S.

- Divide the curved surface S into n pieces that do not step on each other ΔS_i. The symbol for the diameter of the i-th piece is $d_i, i = \overline{1, n}$
- Choose the arbitrary points $M_i\left(x_i, y_i, z_i\right) \in \Delta S_i$. The normal vector offace S at point M_i is $\vec{n}\left(M_i\right) = \left(\cos\alpha_i; \cos\beta_i; \cos\gamma_i\right)$
- Set up totals

$$I_n = \sum_{i=1}^{n} \Delta S_i \vec{F}\left(M_i\right) \cdot \vec{n}\left(M_i\right)$$
$$= \sum_{i=1}^{n} \left[P\left(M_i\right)\cos\alpha_i + Q\left(M_i\right)\cos\beta_i + R\left(M_i\right)\cos\gamma_i\right] \Delta S_i$$

3. Definition of the surface integral of type 2

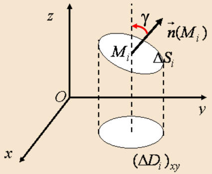
$$I_n = \sum_{i=1}^{n} \left(P\left(M_i\right)\cos\alpha_i + Q\left(M_i\right)\cos\beta_i + R\left(M_i\right)\cos\gamma_i\right)\Delta S_i$$

is called the sum of surface integrals of the second type of the three functions P, Q, R taken on the surface S oriented in $\vec{n}(M)$ with one way to divide and one way to choose $M_i \in \Delta S_i, i = 1, ..., n$.

- If when $n \to \infty$ so that $\max d_i \to 0$ but I_n converges to the number I regardless of the division of S and the choice of $M_i \in \Delta S_i$ then the number I is called the face integral of the second kind of the three functions P, Q, R, taken on the surface $I = \iint\limits_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]dS$.
- The integral surfaces of the second type of the vector field $\vec{F}(P,Q,R)$ pass curvature S in the direction $\vec{n}$ is the first-order surface integral of $\vec{F}$.

3. Definition of the surface integral of type 2

$$I = \iint\limits_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]dS$$



Calling $(\Delta D_i)_{xy}, (\Delta D_i)_{yz}, (\Delta D_i)_{zx}$
is in turn the projection
of ΔS_i onto the coordinate plane Oxy, Oyz, Ozx
$(\Delta D_i)_{xy} = \Delta S_i \cos\gamma_i \Rightarrow \cos\gamma dS = dxdy$
$(\Delta D_i)_{yz} = \Delta S_i \cos\alpha_i \Rightarrow \cos\alpha dS = dydz$
$(\Delta D_i)_{zx} = \Delta S_i \cos\beta_i \Rightarrow \cos\beta dS = dxdzdz$
Therefore, the face integral of the second kind of the functions P, Q, R on the surface S can sign
$I = \iint\limits_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$

IV. Surface integral of type 2

Remark 1.

- It has been shown that, if the face S is oriented, smooth or fragmented smooth and the functions P, Q, R, are continuous on S then the second-order surface integral of P, Q, R exists.
- If the direction of the surface integral is changed, then the surface integral of second-order changes sign.
- The surface integrals of the second-order have the same properties as integrals dual.
- The flux of the vector field $\vec{F}(P, Q, R)$ past the curved surface S orientation is calculated by the formula $\Phi = \iint\limits_{S} Pdydz + Qdzdx + Rdxdy.$
- Assume that the liquid flows through the surface S with velocity $\vec{v}(M)$. Then the flux of the vector field $\vec{v}(M)$ overtaking S is amount of liquid flowing through S in a unit of time.

a. How to calculate the surface integrals of type 2

Theorem 1.

Suppose $R(x, y, z)$ is continuity on a smooth S-oriented surface for by equation $z = z(x, y), (x, y) \in D \subset (Oxy)$. Then

$$\iint_S R(x, y, z)dxdy = \iint_D R(x, y, z(x, y))dxdy$$

if the surface integral of the second-order is taken over the surface S

$$\iint_S R(x, y, z)dxdy = -\iint_D R(x, y, z(x, y))dxdy$$

if the surface integral of the second-order is taken over the surface S.

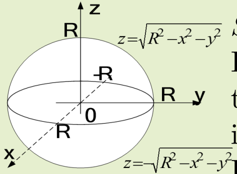
IV. Surface integral of type 2

Similarly, we also have

$$\iint\limits_S P(x,y,z)dydz = \begin{cases} \iint\limits_{D_{yz}} P(x(y,z),y,z)dydz & \text{khi } \cos\alpha \geq 0 \\ -\iint\limits_{D_{yz}} P(x(y,z),y,z)dydz & \text{khi } \cos\alpha \leq 0 \end{cases}$$

$$\iint\limits_S Q(x,y,z)dzdx = \begin{cases} \iint\limits_{D_{=x}} Q(x,y(z,x),z)dzdx & \text{khi } \cos\beta \geq 0 \\ -\iint\limits_{D_x} Q(x,y(z,x),z)dzdx & \text{khi } \cos\beta \leq 0 \end{cases}$$

Example 1.



Calculate $I = \iint\limits_S zdxdy$, where

S is the outside of the sphere $x^2+y^2+z^2=R^2$
Divide the sphere into the upper half S_+ and the bottom half S_- there is a way program in turn is $z=\sqrt{R^2-x^2-y^2}$ and $z=-\sqrt{R^2-x^2-y^2}$
Projecting the halves of the sphere on Oxy, then we get $D: x^2+y^2 \le R^2$

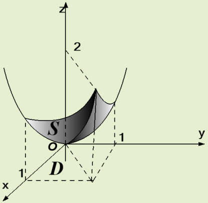
$$I = \iint\limits_{S_+} zdxdy + \iint\limits_{S_-} zdxdy = 2\iint\limits_D \sqrt{R^2-x^2-y^2}dxdy$$

$$I = 2\int\limits_0^{2\pi} d\varphi \int\limits_0^R \sqrt{R^2-r^2}\, rdr = \frac{4}{3}\pi R^3$$

Example 2.

Find the flux of the vector field over the top of the curved surface $z = x^2 + y^2, -1 \le x \le 1, -1 \le y \le 1$

$$\Phi = \iint\limits_S z dydz + x^2 dxdy$$



Due to the S-curve against commensurate with the coordinate planes Oyz degrees so

$$\iint\limits_S z dydz = 0$$

So that

$$\Phi = \iint\limits_S x^2 dxdy = \iint\limits_D x^2 dxdy; \quad D \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \end{cases}$$

$$\Phi = \iint\limits_D x^2 dxdy = \int\limits_{-1}^{1} x^2 dx \int\limits_{-1}^{1} dy = \frac{4}{3}.$$

b. Convert to surface integrals of type 1

Suppose that the P(x,y,z) , Q(x,y,z) R(x,y,z) are integrable on the surface S has a normal vector $\vec{n} = (\cos\alpha; \cos\beta; \cos\gamma)$. Then

$$I = \iint\limits_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma] dS$$

$$= \iint\limits_S \vec{F}(P,Q,R).\vec{n}.dS$$

c. Ostrogradsky- Gauss' formula (O-G)

Let V be a simple solid region and let S be the boundary surface of V, given with positive (outward) orientation. Let P, Q, R be to have continuous partial derivatives on an open region that contains S. Then

$$\iint\limits_S Pdydz + Qdzdx + Rdxdy = \iiint\limits_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz.$$

Example 4.

Calculate $I = \iint\limits_S x^3dydz + y^3dzdx + z^3dxdy$, where S is the outer surface of the sphere $x^2 + y^2 + z^2 = R^2$.

IV. Surface integral of type 2

Remark 3.

- Consider $P = x, Q = y, R = z$, we get the formula calculate the volume of the body V is $V = \frac{1}{3}\iint_S xdydz + ydzdx + zdxdy$, where S is oriented outside the domain V.
- It can be considered that the Ostrogradsky-Gauss'formula is extended Green's formula from two-dimensional space to three-dimensional. Sometimes integrating on a non-closed surface S, we can add a curved surface somewhere to apply the Ostrogradsky-Gauss' formula.
- If $\vec{F} = (P, Q, R)$ is a vector field whose component functions have continuous partial derivatives on an open region that contains S, then the Ostrogradsky-Gauss' formula can be written in the form

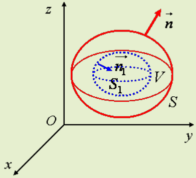
$$\iint\limits_S \vec{F} \cdot \vec{n} \cdot dS = \iiint_V \operatorname{div} \vec{F}.\, dxdydz$$

Corollary.

Assume the functions P, Q, R are continuous partial derivatives in the $V \subset \mathrm{R}^3$ whose outer boundary is a closed surface S ,the inner boundary is the closed surface S_1 which is smooth each piece. Then

$$\iint\limits_{S} Pdydz + Qdzdx + Rdxdy - \iint\limits_{S_1} Pdydz + Qdzdx + Rdxdy$$

$$= \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz.$$

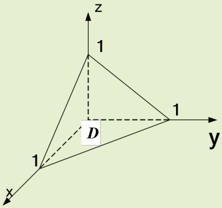


Especially, if $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$, then

$$\iint\limits_{S} Pdydz+Qdzdx+Rdxdy = \iint\limits_{S_1} Pdydz+Qdzdx+R$$

Example 5.

Calculate $I = \iint_S xzdydz + yxdzdx + zydxdy$ take the outside of the surface S as the boundary of the pyramid



$$x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$$

Applying the Odstrogradsky-Gauss' formula

$$I = \iiint\limits_V (z + x + y)dxdydz$$

$$I = \iint\limits_D dxdy \int\limits_0^{1-x-y} (x + y + z)dz$$

$$I = \iint\limits_D dxdy \left((x + y)z + \frac{1}{2}z^2\Big|_{z=0}^{1-x-y}\right) = \int\limits_0^1 dx \int\limits_0^{1-x} \frac{1}{2}\left[1 - (x + y)^2\right] dy$$

$$= \frac{1}{2}\int\limits_0^1 dx \left(y - \frac{1}{3}(x + y)^3\Big|_{y=0}^{y=1-x}\right) = \frac{1}{2}\int\limits_0^1 \left(1 - x - \frac{1}{3} + \frac{1}{3}x^3\right) dx = \frac{1}{8}$$

Example 6.

Calculating the flux of the vector field $\vec{F} = \frac{q\vec{r}}{r^3}$ across the surface $x^2 + y^2 + z^2 = R^2$ in which q is the charge at the root of the coordinates, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$P = q\frac{x}{r^3}, Q = q\frac{y}{r^3}, R = q\frac{z}{r^3}, \forall(x, y, z) \neq (0, 0, 0)$$

Note: We can not apply the Odstrogradsky-Gauss formula in the sphere

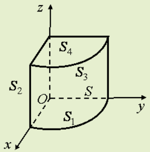
$$\Phi = q \iint\limits_S \frac{1}{r^3}(xdydz + ydzdx + zdxdy)$$

However, the Odstrogradsky-Gauss' formula can be applied to the integral

$$\Phi = \frac{q}{R^3} \iint\limits_S xdydz + ydzdx + zdxdy$$

$$= \frac{q}{R^3} \iiint_V 3dxdydz = \frac{q}{R^3} \cdot 3 \cdot \frac{4}{3}\pi R^3 = 4\pi q.$$

Example 7.



Calculating the flux of the vector field $\vec{F}\left(x^3, y^3, z^3\right)$ through the outside of the cylindrical part $x^2 + y^2 = R^2, x \geq 0, y \geq 0, 0 \leq z \leq h$.

$$\Phi = \iint\limits_S x^3 dydz + y^3 dxdz + z^3 dxdy$$

$$\Phi_k = \iint\limits_{S_k} x^3 dydz + y^3 dxdz + z^3 dxdy, \quad \Phi_1 = \Phi_2 = \Phi_3 = 0$$

$$\Phi_4 = \iint\limits_D h^3 dxdy = h^3 \frac{\pi R^2}{4}; D = \left\{(x, y) : x^2 + y^2 \leq R^2, x \geq 0, y \geq 0\right\}.$$

Example 7.

Applying the Odstrogradsky - Gauss' formula, we have

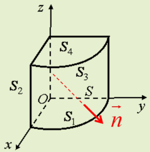
$$\Phi + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = 3 \iiint \left(x^2 + y^2 + z^2\right) dxdyd,$$

$$\iiint_V \left(x^2 + y^2 + z^2\right) dxdydz = \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{R} rdr \int\limits_0^{h} \left(r^2 + z^2\right) dz$$

$$= \frac{\pi h R^4}{8} + \frac{\pi h^3 R^2}{12}$$

$$\Phi = 3\left(\frac{\pi h R^4}{8} + \frac{\pi h^3 R^2}{12}\right) - h^3 \frac{\pi R^2}{4} = \frac{3\pi h R^4}{8}.$$

Example 8.



Calculating the flux of the vector field $\vec{F}\left(x^3, y^3, z^3\right)$ through the outside of the cylindrical part $x^2 + y^2 = R^2, x \geq 0, y \geq 0, 0 \leq z \leq h$

We can calculate directly

$$\Phi = \iint\limits_S x^3 dydz + y^3 dxdz + z^3 dxdy$$

$$\vec{n} = \left(\frac{x}{R}, \frac{y}{R}, 0\right) \Rightarrow \iint\limits_S z^3 dxdy = 0$$

$$\iint\limits_S y^3 dxdz = \iint\limits_D \left(\sqrt{R^2 - x^2}\right)^3 dxdz$$

$$\iint\limits_D \left(\sqrt{R^2 - x^2}\right)^3 dxdz = \int\limits_0^R \left(\sqrt{R^2 - x^2}\right)^3 dx \int\limits_0^h dz = h \int\limits_0^R \left(\sqrt{R^2 - x^2}\right)^3 dx$$

Example 8.

Set $x = R\sin t$, we obtain

$$h\int_0^R \left(\sqrt{R^2 - x^2}\right)^3 dx = hR^4 \int_0^{\pi/2} \cos^4 t dt = \frac{3\pi hR^4}{16}.$$

Thus

$$\iint_S x^3 dydz = \frac{3\pi hR^4}{16} \Rightarrow \Phi = \frac{3\pi hR^4}{8}$$

d. Stokes' formula

The Stokes' formula extends Green's formula, which is the relationship between the second-order line integral in space and the second-order surface integral.

Theorem 3 (Stokes' theorem)

Assuming that the segmented smooth, oriented S-curve has the boundary of the segmented smooth L and
Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let the functions P, Q, and R be continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C Pdx + Qdy + Rdz =$$

$$\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Remark 2.

- When substituting $z = 0, R(x, y, z) = 0$ into the Stokes formula, we get the Green's formula.

-Let the vector field $\vec{F} = (P, Q, R)$ and

$$\operatorname{rot} \vec{F} = [\vec{\nabla}; \vec{F}] = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right),$$

The Stokes' formula can be written in the form

$$\oint_L Pdx + Qdy + Rdz = \iint_S \operatorname{rot} \vec{F} \cdot n \cdot dS = \iint_S \begin{vmatrix} dydz & dxdz & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Theorem 4 (Theorem of equivalence statements)

Assume the functions P, Q, R are continuous partial derivatives in the simple domain V. Then the following propositions are similar.

(1) $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \forall (x, y, z) \in V.$

(2) $\iint\limits_{L} Pdx + Qdy + Rdz = 0, L$ is any closed curve in the domain V.

(3) $\int\limits_{\overset{\frown}{AB}} Pdx + Qdy + Rdz$, where $\overset{\frown}{AB} \subset V$ does't depend on the arc form $\overset{\frown}{AB}$.

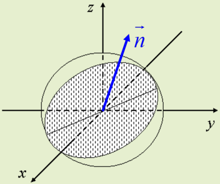
(4) Expression $Pdx + Qdy + Rdz$ is the total differential of some function $u(x, y, z)$ on the domain V and

$$\int\limits_{\overset{\frown}{AB}} Pdx + Qdy + Rdz = u(B) - u(A)$$

Example 9.

Calculate $I = \oint\limits_C ydx + zdy + xdz$, where C is the circle, intersection of the sphere $x^2 + y^2 + z^2 = R^2$ and plane $x + y + z = 0$ and direction of C is counterclockwise if looking towards $z > 0$.

Solution



The plane $x + y + z = 0$ passes through the center of the sphere. So the intersection is the great circle. Take the circle as a curved surface S with boundary C. The direction cosines of $\overrightarrow{n}$ oriented in the direction of C is $\vec{n} = (1, 1, 1)$. Apply Stokes' formula with $\overrightarrow{n_0} = \frac{1}{\sqrt{3}}(1, 1, 1)$

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = (-1, -1, -1)$$

$$I = \iint ydx + zdy + xdz = \iint \operatorname{rot} \vec{F} \cdot \overrightarrow{n_0} \cdot dS = -\sqrt{3} \iint dS = -\sqrt{3}\pi R^2.$$

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Chapter 4: DIFFERENTIAL EQUATIONS

CALCULUS 2

Faculty of Fundamental Science 1

Hanoi - 2022

Outline of differential equations

4.1.1. Definition

Definition 1

A differential equation is an equation involving an unknown function and its derivatives of the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \text{ or } F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0,$$

where x is an independent variable, $y = y(x)$ is the function to find, $y', y'', ..., y^{(n)}$ are the derivatives of the function must find.
The order of a differential equation is the largest derivative present in the differential equation.

Example 1

$y' - x^4y^3 = 5, \; (x^3 + y^2)dx - (x^2 + y^2)dy = 0$ are first-order differential equations.
$y'' - 2x^3(y')^3 = 5$ is a second-order differential equation.

4.1.1. Definition

A differential equation is called an n th-order linear differential equation if the function F is first order with respect to $y, y', \ldots, y^{(n)}$, that is, the equation has the form:
$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x)$, where
$a_1(x), \ldots, a_n(x),\ f(x)$ are continuous functions on (a, b).
If $f(x) \equiv 0, \forall x \in (a, b)$ then it is called an n-th order linear homogeneous differential equation.
If $f(x) \neq 0,\ x \in (a, b)$ then it is called a non-homogeneous.

Example 2

$y'' - x^2y = 5x^2 + 1$ is a second-order linear non-homogeneous differential equation and $y' - x^2y = 0$ is a first-order linear homogeneous differential equation.

4.1.2.Solution

Definition 2

A solution of a differential equation in the unknown function y and the independent variable x on the interval J is a function y(x) that satisfies the differential equation identically for all x in J.

+) The solution is an explicit function $y = y(x, C_1, C_2, ..., C_n)$ depends on the constants $C_1, C_2, ..., C_n$ is called the general solution.

Example 3

The function $y(x) = C_1 \sin 2x + C_2 \cos 2x$, where C_1 and C_2 are arbitrary constants, is a general solution of $y^{''} + 4y = 0$ in the interval $(-\infty, +\infty)$.

+) The solution is the implicit function $\Phi(x, C_1, C_2, ..., C_n) = 0$ depends on which constants $C_1, C_2, ..., C_n$ are called is the general integral.

Example 4

The function $2e^{y^3} + xy^2 = C$ with C is arbitrary constants, is a general integral of $6y'ye^{y^3} + y^2 + 2xy'y = 0$.

+) If we give the constants $C_1, C_2, ..., C_n$ determined values, then the general solution (general integral) is called a particular solution (particular integral).

4.2.1 Introduction to first-order differential equation

Standard form for a first-order differential equation in the unknown function $y(x)$ is

$$F(x, y, y') = 0 \text{ or } F(x, y, \frac{dy}{dx}) = 0 \text{ or } M(x, y)dx + N(x, y)dy = 0 \quad (1)$$

If from (1) we can solve for y', then the first-order differential equation has been solved for the derivative:

$$y' = f(x, y) \quad (2)$$

Example 1

$y' = 3x^5y^3 + 2, \quad (2x - 3y)dx - (2x^2 - y)dy =$ are the first-order differential equations.

Cauchy-Peano's theorem (Existence and uniqueness theorem)

Consider the differential equation (2): $y' = f(x, y)$ and $M_0(x_0, y_0) \in D \subset \mathbb{R}^2$.

Theorem 4.1 If $f(x, y)$ is continuous on the domain D in the plane of Oxy, then there exists a solution $y = y(x)$ in the neighborhood x_0 satisfy

$$y_0 = y(x_0) \quad (3)$$

In addition, if $\dfrac{\partial f}{\partial y}(x, y)$ is also continuous on domain D, then the found solution is unique.

The problem of finding a solution of a differential equation satisfying the condition (3) is called a Cauchy problem. Condition (3) is called the initial condition.

4.2.2 Separable equations

Definition 1

Consider a differential equation in differential form (1). If $M(x, y) = f_1(x)$ (a function only of x) and $N(x, y) = f_2(y)$ (a function only of y), differential equation is separable, or has its variables separated.

Solution

The solution to the first-order separable differential equation $f_1(x)dx + f_2(y)dy = 0$ (1.1) is $\int f_1(x)dx + \int f_2(y)dy = C$ (1.2) where C represents an arbitrary constant.

Example 2

Solve the equation: $\dfrac{dx}{dy} = \dfrac{3x^2 + 1}{2y}$. This equation may be rewritten in the differential form $(3x^2 + 1)dx = 2ydy$. Its solution is $\int(3x^2 + 1)dx - \int 2ydy = C$ or $x^3 + x - y^2 = C$.

4.2.3 Homogeneous equations

Definition 2

A differential equation in standard form $y' = f(x, y)$ (1.3) is homogeneous if $f(tx, ty) = f(x, y)$ for every real number $t \neq 0$.
Consider $x \neq 0$. Then, we can write
$f(x, y) = f(x, x\frac{y}{x}) = f(1, \frac{y}{x}) := g(\frac{y}{x})$ for a function g depending only on the ratio $\frac{y}{x}$.

Solution

The homogeneous differential equation can be transformed into a separable equation by making the substitution $y = xu$ along with its corresponding derivative $y' = u + xu'$. This can be rewritten as
$\frac{du}{g(u) - u} = \frac{dx}{x}$ if $g(u) - u \neq 0$. The resulting equation in the variables u and x is solved as a separable differential equation.

4.2.3 Homogeneous equations

Example 3

Solve $y' = \dfrac{y+x}{x}$ for $x \neq 0$.
This differential equation is not separable. Instead it has the form $y' = f(x,y)$, with $f(x,y) = \dfrac{y+x}{x}$, where $f(tx,ty) = f(x,y)$, so it is homogeneous.

Example 4

Integral equation $(y-x+1)dx = (x+y+3)dy$.
Solution: $\dfrac{dy}{dx} = \dfrac{y-x-1}{x+y+3}$.

Definition 3

A first-order linear differential equation has the form

$$y' + p(x)y = q(x), \quad (1.4)$$

where $p(x),\ q(x)$ are continuous on (a, b). In other words, a linear differential equation of the first order is a differential equation in which the function to be found and its derivative are both in first order form.
- If $q(x) \neq 0,\ x \in (a, b)$ then (1.4) is called a non-homogeneous linear differential equation.
- If $q(x) \equiv 0,\ \forall x \in (a, b)$ then call it a homogeneous linear differential equation.

4.2.4 Linear equations

Method of solutions:

The general solution for equation (1.4) is

$$y = e^{-\int p(x)dx}\left(C + \int q(x)e^{\int p(x)dx}dx\right),$$

where C represents an arbitrary constant.

Example 5

Find the solution of the given initial value problem
$y' - \frac{3}{x}y = x^3, y(1) = 2.$

Example 6

Solve $e^y dx + (xe^y - 1)dy = 0.$
Solution: $x' - x = e^{-y}.$

4.2.5 Bernoulli equations

Definition 4

A Bernoulli differential equation has the form

$$y' + p(x)y = q(x)y^{\alpha}, \quad (1.5)$$

where α is a real number $\alpha \neq 0,\ \alpha \neq 1$.

Solution

If $\alpha > 0$, then $y = 0$ is a solution of (1.5). Otherwise, if $\alpha < 0$, then the condition is $y \neq 0$. In both cases, we now find the solutions $y \neq 0$. To do this we divide both sides by y^{α} to obtain $y^{-\alpha}y' + p(x)y^{1-\alpha} = q(x)$. The substitution $z = y^{1-\alpha}$ now transforms (1.5) into a linear differential equation in the unknown function $z = z(x)$.

Example 7

Solve $y' + xy = xy^2$.

Solution: This equation is not linear. It is, however, a Bernoulli differential equation having the form of equation (1.5) with $p(x) = q(x) = x$, and $\alpha = 2$. First, we can see that $y = 0$ is a solution of the equation. We now find the solution $y \neq 0$. To do so, we make the substitution: $z = y^{1-2} = y^{-1}$, from which follow $y = 1/z$ and $y' = -\frac{z'}{z^2}$. Substituting these equations into the given differential equation, we obtain the equation $z' - xz = -x$ which is linear for the unknown function $z(x)$.

Example 8

Solve $y' + y = e^{\frac{x}{2}}\sqrt{y}$.

4.2.6 Exact equations

Definition 5

A differential equation in differential form

$$M(x, y)dx + N(x, y)dy = 0 \quad (1.6)$$

is exact if there exists a function $g(x, y)$ such that
$dg(x, y) = M(x, y)dx + N(x, y)dy$ or $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}, \forall (x, y) \in D \subset \mathbb{R}^2 \quad (1.7)$.

Solution

To solve equation (1.6), assuming that it is exact, first solve the equations $\dfrac{\partial g(x, y)}{\partial x} = M(x, y)$, $\dfrac{\partial g(x, y)}{\partial y} = N(x, y)$ for g(x, y). We have $g(x, y) = \int\limits_{x_0}^{x} M(x, y)dx + \int\limits_{y_0}^{y} N(x_0, y)dy$. The solution to (1.6) is then given implicitly by $g(x, y) = C$, where $(x_0, y_0) \in D, C$ represents an arbitrary constant.

Example 9

Solve $2xydx + (1 + x^2)dy = 0$.

Solution: This equation has the form of equation (1.6) with $M(x, y) = 2xy$ and $N(x, y) = 1 + x^2$ are determined on $D = \mathbb{R}^2$. Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y} = 2x$, the differential equation is exact. The solution to the differential equation, which is given implicitly by (1.7) as $g(x, y) = C$, is $x^2y + y = C$.

Example 10

Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

Integrating Factors

In general, equation (1.6) is not exact. Occasionally, it is possible to transform (1.6) into an exact differential equation by a judicious multiplication. A function $I(x, y)$ is an integrating factor for (1.6) if the equation

$$I(x, y)\big(M(x, y)dx + N(x, y)dy\big) = 0 \quad (1.8)$$

is exact. Some of the following special form integral factors:

1. if $\dfrac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv g(x)$, a function of x alone, then $I(x, y) = e^{\int g(x)dx}$.
2. item if $\dfrac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv g(y)$, a function of y alone, then $I(x, y) = e^{-\int g(y)dy}$.
3. item if $M = yf(x, y)$ and $N = xg(x, y)$, then $I(x, y) = \frac{1}{xM - yN}$.

Remark

In general, integrating factors are difficult to uncover. If a differential equation does not have one of the forms given above, then a search for an integrating factor likely will not be successful, and other methods of solution are recommended.

Example 11

Solve $ydx - xdy = 0$.

Solution: This equation is not exact. It is easy to see that an integrating factor is $I(x,y) = \dfrac{1}{x^2}$. Therefore, we can rewrite the given differential equation as $\dfrac{ydx - xdy}{x^2} = 0$ which is exact.

4.3. Second-order differential equation

4.3.1 Introduction to second-order differential equation

Definition 1. A second-order differential equation has the form

$$F(x, y, y', y'') = 0 \text{ or } F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0, \quad (4.3.1a)$$

If from (4.3.1) we can solve for y'', then the second-order differential equation has been solved for the derivative:

$$y'' = f(x, y, y'), \quad (4.3.1b)$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we will use x instead.

Example 1

$y'' = x^2y^5 + 2y^4, \quad (2x - 3y)y'' - (2x^2 - y)\sqrt[3]{y}'' = x^2y'$ are the

4.3.1 Introduction to second-order differential equation

Theorem 1 (Cauchy-Peano's theorem (Existence and uniqueness theorem))

Consider the differential equation (4.3.1b): $y^{''} = f(x, y, y')$ and $M_0(x_0, y_0, y_0') \in V \subset \mathbb{R}^3$.
If $f(x, y)$ is continuous on the domain V in the plane of $Oxyy'$, then there exists a solution $y = y(x)$ in the neighborhood x_0 satisfy

$$y_0 = y(x_0),\ y_0' = y'(x_0), \qquad (4.3.1c)$$

In addition, if $\dfrac{\partial f}{\partial y}(x, y, y'),\ \dfrac{\partial f}{\partial y'}(x, y, y')$ are also continuous on domain V, then the find solution is unique.

4.3.2 Second-order linear differential equations

Definition 2

A second-order differential equation is called linear if it can be written in the form

$$y'' + p(x)y' + q(x)y = f(x), \quad (4.3.2a)$$

where $p(x),\ q(x),\ f(x)$ are continuous on (a, b). In other words, a linear differential equation of the first order is a differential equation in which the function to be found and its derivative are both in first order form.
- If $f(x) \neq 0,\ x \in (a, b)$ then the second-order linear equation (4.3a) is said to be non-homogeneous.
- If $f(x) \equiv 0,\ \forall x \in (a, b)$ then the second-order linear equation (4.3a) is said to be homogeneous has form $y'' + p(x)y' + q(x)y = 0,\ (4.3.2b)$

Example 2

The following equations: $y'' + 2y = e^x \sin x$ and $y'' + 3xy' + 5y = 0$ are examples of nonhomogeneous and homogeneous second-order linear equations, respectively.

Theorem 3

If y_1 and y_2 are two solutions of the differential equation $(4.3.2b)$, then the linear combination $y = C_1 y_1 + C_2 y_2$ is also a solution for any values of the constants C_1 and C_2.

Proof

$y' = C_1 y_1' + C_2 y_2'$ and $y'' = C_1 y_1'' + C_2 y_2''$. We obtain

$$(C_1y_1 + C_2y_2)'' + p(x)\left(C_1y_1 + C_2y_2\right)' + q(x)\left(C_1y_1 + C_2y_2\right)$$
$$+\, C_1\left[y_1'' + p(x)y_1' + q(x)y_1\right] + C_2\left[y_2'' + p(x)y_2' + q(x)y_2\right] \equiv 0.$$

4.3.2 Second-order linear differential equations

The two functions $y_1(x)$ and $y_2(x)$ are called linearly independent on (a, b), if $\alpha_1 y_1(x) + \alpha_2 y_2(x) = 0$ for all $x \in (a, b)$ implies $\alpha_1 = \alpha_2 = 0$, and we call $y_1(x),\ y_2(x)$ linearly dependent on (a, b) if there exist $\alpha_1,\ \alpha_2$ not both zero such that $\alpha_1 y_1(x) + \alpha_2 y_2(x) = 0,\ t \in (a, b)$.

The $y_1(x) = e^x$ and $y_2(x) = e^{-2x}$ are linearly independent, because they are not proportional.

Wronski determinant (or Wronskian)

The Wronski determinant (or Wronskian) of the two solutions $y_1(x), y_2(x)$ of the equation $(4.3.2b)$ is defined by

$$W\left[y_1, y_2\right] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Theorem 4

The two solutions $y_1(x)$ and $y_2(x)$ of Eq. (4.3.2b) are linearly dependent on (a, b) if and only if their Wronskian $W[y_1, y_2]$ is zero at for all points $x \in (a, b)$. That is,

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \equiv 0, \ \forall x \in (a, b).$$

If there exists $x_0 \in (a, b)$ such that $W[y_1(x_0), y_2(x_0)] \neq 0$, then the $y_1(x), y_2(x)$ are independent.
On the contrary, if the two solutions $y_1(x), y_2(x)$ of Eq. (4.3.2b) are linearly independent on (a, b) then

$$W(x) = W[y_1, y_2] \neq 0, \forall x \in (a, b).$$

Theorem 5 (Structure of solutions to homogeneous equations)

If there exists two linearly independent solutions y_1, y_2 on (a, b) of Eq.(4.3.2b), then the general solution (the set of all solutions) of Eq.(4.3.2b) is

$$y = C_1 y_1 + C_2 y_2,$$

where C_1, C_2 are arbitrary constants.

Example

Solve the following differential equation: $y'' - 3y' + 2y = 0$, knowing that $y_1 = e^x$ and $y_2 = e^{2x}$ are solutions.

4.3.2 Second-order linear differential equations

Theorem 6 (Liouville's theorem)

If a nontrivial solution y_1 is known, then we can find the solution y_2 linearly independent with y_1 of Eq. (4.3.2b) by formula

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int p(x)dx} dx.$$

Example 8

Solve the differential equation: $y'' + \frac{2}{x}y' + y = 0$ know a solution $y_1 = \dfrac{\sin x}{x}$.

Solution: $y_2 = \dfrac{\sin x}{x} \displaystyle\int \frac{x^2 \cdot e^{-\int \frac{2}{x} dx}}{\sin^2 x} dx = \frac{\sin x}{x} \int \frac{x^2 \cdot e^{-2\ln x}}{\sin^2 x} dx =$
$\dfrac{\sin x}{x} \displaystyle\int \frac{dx}{\sin^2 x} = \frac{\sin x}{x}(-\cot x) = -\frac{\cos x}{x}.$
Therefore, the general solution equqtion is $y = \frac{1}{x}\left(C_1 \sin x + C_2 \cos x\right).$

Example 9

Solve the equation $x^2(\ln x - 1)y'' - xy' + y = 0$ know that it has a solution that is a power function.

Solution: Put $y_1 = x^a$ in the equation, we have

$$x^2(\ln x - 1)\alpha(\alpha-1)x^{\alpha-2} - \alpha x^\alpha + x^\alpha = 0, \forall x \in (a,b)$$
$$\Rightarrow \alpha(\ln x - 1)(\alpha - 1) - \alpha + 1 = 0, \forall x \in (a,b)$$
$$\Rightarrow \begin{cases} \alpha(\alpha-1) = 0 \\ -\alpha + 1 = 0 \end{cases} \Rightarrow \alpha = 1 \Rightarrow y_1 = x$$
$$y_2 = x\int \frac{e^{\int \frac{x\,dx}{x^2(\ln x - 1)}dx}}{x^2} = x\int \frac{e^{\int \frac{d\ln x}{\ln x - 1}}}{x^2}dx = x\int \frac{e^{\ln(\ln x - 1)}}{x^2}dx$$
$$= x\int \frac{\ln x - 1}{x^2}dx = x\left[-\frac{1}{x}(\ln x - 1) + \int \frac{dx}{x^2}\right] = -\ln x$$

Therefore, so the general solution is $y = C_1 x + C_2 \ln x$.

4.3.2 Second-order linear differential equations

Theorem 7 (Structure of solutions to non-homogeneous equations)

The general solution of the non-homogeneous equation (4.3.2a) is equal to the general solution $\bar{y}$ of the homogeneous equation (4.3.2b) plus some particular solution y^* of the equation (4.3.2a). That is $y = \bar{y} + y^*$.

Theorem 8 (Principle of superposition of solutions)

Now suppose that $f(x)$ is the sum of two terms, $f(x) = f_1(x) + f_2(x)$, and suppose that y_1^* and y_2^* are solutions of the equations

$$y^{''} + py^{'} + qy = f_1(x), \quad (4.3.2a1)$$

and

$$y^{''} + py^{'} + qy = f_1(x), \quad (4.3.2a2)$$

respectively. Then $y^* = y_1^* + y_2^*$ is a particular solution of the equation (4.3.2a)(i.e. $y^{''} + py^{'} + qy = f_1(x) + f_2(x)$).

4.3.3. Second-order linear differential equations with constant coefficients

Definition 3

The second-order differential equation is called the linear second-order differential equations with constant coefficients of the form

$$y'' + py' + qy = f(x), \quad (4.3.3a)$$

where p, q are real constants, $f(x)$ is continuous on (a, b).
- If $f(x) \equiv 0, \forall x \in (a, b)$ then $(4.3.3a)$ is called the linear second-order differential honogeneous equations with constant coefficients of the form

$$y'' + py' + qy = 0, \quad (4.3.3b).$$

If $f(x) \neq 0, x \in (a, b)$ then $(4.3.3a)$ is called the linear second-order differential non-honogeneous equations with constant coefficients.

4.3.3. Second-order linear differential equations with constant coefficients

a. General solutions of homogeneous equations with constant coefficients

To seek exponential solutions, we suppose that $y = y^{kx}$, where k is a constant.Then it follows that $y' = ke^{kx}$ and $y'' = k^2e^{kx}$. By substituting these expression for y, y', and y'' in Eq. $(4.3.3b)$, we obtain $(k^2 + pk + q)e^{kx} = 0$ or, since e^{kx} is never zero,

$$k^2 + pk + q = 0, \quad (4.3.3c)$$

is called the characteristic equation for the Eq.(4.3.3b).

a.Homogeneous equations with constant coefficients

We now consider solution of the characteristic equation

$$k^2 + pk + q = 0(4.3.3c)$$

1^{st} **Case: Distinct real roots.**

Assuming that the roots of the characteristic equation (4.3.3c) are real and different, let them be denoted by k_1 and k_2, where $k_1 \neq k_2$. Then $y_1 = e^{k_1 x}$ and $y_2 = e^{k_2 x}$ are two linearly dependent solutions of Eq.(4.3.3b). Therefore, we obtain the general solution of Eq.((4.3.3b)):$\bar{y} = C_1 e^{k_1 x} + C_2 e^{k_2 x}$.

Example 12

Find the general solution of $y'' + 5y' + 6y = 0$. The characteristic equation is $k^2 + 5k + 6 = 0$. It has two distinct real roots: $k_1 = -2$ and $k_2 = -3$, then the general solution is $\bar{y} = C_1 e^{-2x} + C_2 e^{-3x}$

a.Homogeneous equations with constant coefficients

2^{nd} **Case: Double real root.**

We consider the second possibility, namely, that the two real roots k_1 and k_2 are equal. This case occurs when the discriminant $\Delta = p^2 - 4q$ is zero, and it follows from the quadratic formula that $k_1 = k_2 = -\frac{p}{2}$.The difficulty is immediately apparent; both roots yield the same solution $y_1 = e^{k_1 x} = e^{-\frac{p}{2}x}$ of the differential equation (4.3.3c). We now find a second solution y_1 which is linearly independent to y_2, we have $y_2 = e^{-\frac{px}{2}} \int \frac{e^{-\int p dx}}{e^{-px}} dx = xe^{-\frac{px}{2}} = x.y_1$. Therefore, the general solution of Eq. (4.3.3b) in this case is $\bar{y} = e^{kx}\left(C_1 + C_2 x\right)$.

Example 13

Solve the differential equation $y'' + 4y' + 4y = 0$, The characteristic equation is $k^2 + 4k + 4 = 0$, which has a double real root $k_1 = k_2 = -2$. Therefore, the general solution of given differential equation is

$$y = \left(C_1 + C_2 x\right) e^{-2x}.$$

a.Homogeneous equations with constant coefficients

3^{rd} **Case: Complex conjugate roots.**

If the roots of characteristic $(4.3.3c)$ are conjugate complex numbers $k = \alpha \pm i\beta$, then the general solution of Eq. $(4.3.3b)$ is

$$\bar{y} = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right),$$

where $y_1 = \frac{e^{\alpha x + i\beta x} + e^{\alpha x - i\beta x}}{2}$, $y_2 = \frac{e^{\alpha x - i\beta x} + e^{\alpha x - i\beta x}}{2i}$.

Example 14

Find the general solution of $y'' - 4y' + 13y = 0$, The characteristic equation is $k^2 - 4k + 13 = 0$, and its roots are $k = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$
Thus, the general solution is $\bar{y} = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)$

$$y(0) = 1 = C_1; y'(0) = 1 = 2C_1 + 3C_2$$

$$\Rightarrow C_2 = -\frac{1}{3} \Rightarrow y = e^{2x}\left(\cos 3x - \frac{1}{3}\sin 3x\right)$$

Remark

- We can find the general solution of the homogeneous coefficient second order linear differential equation by solving the corresponding characteristic equation
- To find particular solution of non-homogeneous second order linear differential equations, we can use the method of variation of the Lagrange constant and the principle of superposition of solutions.

Example 15

Find the general solution of $y'' - y = \frac{e^x}{e^x+1}$.
The characteristic equation is
$k^2 - 1 = 0 \Rightarrow k = \pm 1 \Rightarrow y = C_1 e^{-x} + C_2 e^x$

$\begin{cases} C_1' e^{-x} + C_2' e^x = 0 \\ -C_1' e^{-x} + C_2' e^x = \frac{e^x}{1+e^x} \end{cases} \Rightarrow \begin{cases} C_1' = -\frac{1}{2}\frac{e^{2x}}{1+e^x} \\ C_2' = \frac{1}{2}\frac{1}{e^x+1} \end{cases}$ Thus, we obtain

$$y = \frac{e^{-x}}{2}\left[\ln\left(e^x+1\right) - e^x + C_1\right] + \frac{e^x}{2}\left[x - \ln\left(e^x+1\right) + C_2\right].$$

4.3.3. Second-order linear differential equations with constant coefficients

b. General solutions of non-homogeneous equations

The linear second-order differential non-homogeneous equations with constant coefficient of the special function $f(x)$ has the corresponding particular solutions formula.

FORM 1: $f(x) = e^{\alpha x} P_n(x)$

+ If the constant α is not a root of the characteristic equation, then $(4.3.3a)$ has a particular solution of the form $y^* = e^{\alpha x} Q_n(x)$.

+ If the constant α is a single root of the characteristic equation,then $(4.3.3a)$ has a particular solution of the form $y^* = x e^{\alpha x} Q_n(x)$.

+ If the constant α is the double root of the characteristic equation,then $(4.3.3a)$ has a particular solution of the form

$$y^* = x^2 e^{\alpha x} Q_n(x)$$

b. General solutions of non-homogeneous equations

FORM 2: $f(x) = e^{\alpha x}\left[P_n(x)\cos\beta x + P_m(x)\sin\beta x\right]$

If $\alpha \pm i\beta$ is not a root of the characteristic equation, then $(4.3.3a)$ has a particular solution of the form

$$y^* = e^{\alpha x}\left[Q_k(x)\cos\beta x + R_k(x)\sin\beta x\right]$$

where $Q_k(x), R_k(x)$ are polynomials of degree $k = max(m, n)$.

If $\alpha \pm i\beta$ is a root of the characteristic equation, then $(4.3.3a)$ has a particular solution of the form

$$y^* = xe^{\alpha x}\left[Q_k(x)\cos\beta x + R_k(x)\sin\beta x\right]$$

Example 16

Find the general solution of

$$y'' + 2y' - 3y = e^x x + x^2$$

The characteristic equation is $k^2 + 2k - 3 = 0 \Leftrightarrow \left[\begin{array}{l} k_1 = 1 \\ k_2 = -3 \end{array}\right.$

The general solution of the homogeneous equation is $\bar{y} = C_1 e^{-3x} + C_2 e^x$

+ particular solution $y_1^* = xe^x(ax+b) {y^*}'_1 = e^x \left(ax^2 + bx + 2ax + b\right)$

$${y^*}''_1 = e^x \left(ax^2 + bx + 4ax + 2b + 2a\right) \Rightarrow {y^*}_1 = \frac{x}{8} e^x (x-1).$$

+ particular solution ${y^*}_2 = ax^2 + bx + c \quad {y^*}'_2 = 2ax + b \quad {y^*}''_2 = 2a$

$$\Rightarrow {y^*}_2 = x^2 - 4x + 14.$$

The general solution of the nonhomogeneous equation is

$$y = \bar{y} + {y^*}_1 + {y^*}_2 = C_1 e^{-3x} + C_2 e^x + \frac{x}{8} e^x (x-1) + x^2 - 4x + 14$$

Example 17

Finding solutions to Cauchy's problem

$$y'' - 4y' + 4y = e^{2x}(x+1), y(0) = y'(0) = 1$$

The characteristic equation is $k^2 - 4k + 4 = 0 \Leftrightarrow k_1 = k_2 = 2$ The general solution of the homogeneous equation is $\bar{y} = C_1 + C_2 e^{2x}$
Particular solution:
$y^* = x^2 e^{2x}(ax+b) y^{*\prime} = e^{2x}\left(2ax^3 + 2bx^2 + 3ax^2 + 2bx\right)$

$y^{*\prime\prime} = e^{2x}\left(4ax^3 + 4bx^2 + 12ax^2 + 6ax + 8bx + 2b\right) \Rightarrow y^* = \frac{1}{6}x^2 e^{2x}(x+3)$

$y = \bar{y} + y^* = e^{2x}(C_1 + C_2 x) + \frac{1}{6}x^2 e^{2x}(x+3)$

$y' = e^{2x}(2C_1 + C_2 + 2C_2 x) + \frac{1}{6}e^{2x}\left(2x^3 + 9x^2 + 6x\right)$

$y(0) = C_1 = 1 \Rightarrow y'(0) = 2C_1 + C_2 = 1$

$C_1 = 1, C_2 = -1 \Rightarrow y = e^{2x}(1-x) + \frac{1}{6}x^2 e^{2x}(x+3)$

Example 18

Find the general solution of $y'' + y' = xcosx$.

The characteristic equation is $k^2 + k = 0 \Leftrightarrow \left[\begin{array}{l} k_1 = 0 \\ k_2 = -1 \end{array} \right.$

The general solution of the homogeneous equation is $\bar{y} = C_1 + C_2 e^{-x}$.

Particular solution: $y^* = (ax + b)\cos x + (cx + d)\sin x$

Substituting into the differential equation we have

$((c-a)x+a+2c+d-b)\cos x+(-(a+c)x+c-2a-b-d)\sin x = x\cos x.$

$$\begin{cases} c - a = 1 \\ a + 2c + d - b = 0 \\ a + c = 0 \\ c - 2a - b - d = 0 \end{cases} \Rightarrow a = -\frac{1}{2}, b = 1, c = \frac{1}{2}, d = \frac{1}{2}$$

Thus, the general solution is

$$y = C_1 + C_2 e^{-x} - \frac{1}{2}(x - 2)\cos x + \frac{1}{2}(x + 1)\sin x.$$

Example 19

Find the general solution of $y'' + 2y' + 2y = e^{-x}(1 + \sin x)$.
The characteristic equation is $k^2 + 2k + 2 = 0 \Leftrightarrow k = -1 \pm i$.
The general solution of the homogeneous equation is

$$\bar{y} = e^{-x}\left(C_1 \cos x + C_2 \sin x\right).$$

The particular solution of $y'' + 2y' + 2y = e^{-x} \sin x$ has the form $y_1^* = xe^{-x}(a \cos x + b \sin x)$. We have

$$2b \cos x - 2a \sin x = \sin x \Rightarrow b = 0, a = -\frac{1}{2} \Rightarrow y_1^* = -\frac{xe^{-x}}{2} \cos x.$$

The particular solution of $y'' + 2y' + 2y = e^{-x}$ has form $y_2^* = ce^{-x}$. We get $y_2^* = e^{-x}$.
Thus, the general solution is

$$y = \bar{y} + y_1^* + {y^*}_2 = e^{-x}\left(C_1 \cos x + C_2 \sin x\right) + e^{-x}\left(1 - \frac{x}{2} \cos x\right).$$